Consensus on Social Influence Network Model

Zhenpeng Li[1] and Xijin Tang[2(✉)]

[1] Department of Applied Statistics, Dali University, Dali 671003, China
lizhenpeng@amss.ac.cn
[2] Academy of Mathematics and Systems Sciences,
Chinese Academy of Sciences, Beijing 100190, China
xjtang@amss.ac.cn

Abstract. Many studies show that opinions formation displays multiple patterns, from consensus to polarization. Under the framework of the social influence network by Friedkin and Johnsen (1999) and based on random walk on graph, we rigorously prove that for a social group influence system, with static social influence structure, the group consensus is almost a quite certain result. In addition, we prove the lower bounds on the convergence time m for random walk P^m to be close to its final average consensus (wisdom group decision making) state, given an arbitrary initial opinions profile vector and one small positive error ϵ. Although our explanations are purely based on mathematic deduction, it shows that the latent social influence structure is the key factor for the persistence of disagreement and formation of opinions convergence or consensus in the real world social group.

Keywords: Social influence network theory · Random graph · Opinions dynamics

1 Introduction

Originally from decentralized decision making, consensus problems have an old history, such as the models introduced in DeGroot(1974)[1], Friedkin and Johnsen (1999)[2] and Friedkin (2011)[3]. From social psychological point of view, this line of research began with French's formal theory of social power[4], a simple model of collective opinion formation in a network of interpersonal influencing social group. As a step forward, Friedkin presented the social influence network theory, which considered both cognitive and structural aspects, and focused on the contributions of networks of interpersonal influence to the formation of interpersonal agreements and group consensus.

Over the past few years, models of the convergence of opinion or consensus problem in social systems have been the subjects of a considerable amount of recent attention in the fields such as motion coordination of autonomous agents [5,6], distributed computation in control theory [5,7,8], randomized consensus algorithms [9,10], and sensor networks about data fusion problems [11]-[15].

M.T. Thai et al. (Eds.): CSoNet 2015, LNCS 9197, pp. 74–83, 2015.
DOI: 10.1007/978-3-319-21786-4_7

Most of the growing interests in consensus problems (both algorithms and practical applications) are based on probabilistic settings. This might be due to the unpredictability of the environment where the communication between agents occurs [9], and the random characteristics of influences or interactions among agents in systems (man made or social systems).

Recently, the study of opinion dynamics has started to attract the attention of the control community, who with the bulk of motivation have developed about methods to approximate and stabilize consensus, synchronization, and other coherent states. However, comparing with many man-made or engineering systems, social systems do not typically exhibit a consensus of opinions, but rather a persistence of disagreement, i.e. polarization patterns. The ubiquitous group polarization phenomena can be observed from political election to carbon dioxide emissions debate[16]. In a social system, the difficulty in arriving at a collective consensus state roots in the fact that the process of opinion formation can rarely be reduced to accepting or rejecting the consensus of others, as exemplified by Arrow's dilemma of social choice [17]. On the contrary, in most cases individuals construct their options in a complex interpersonal environment or with their prior identities (e.g. prior beliefs, prejudices and social identities etc.), their views are often in a state of disagreement or not easily changed, due to opinion-dependent limitations in the network connectivity and obstinacy of the agents as pointed in Ref. [18]. This phenomenon shows the complexities of social control in social economic systems.[1]

Consensus as one of the important and regular group opinions dynamic pattern is generally observed in a relative smaller group discussion and barging process. Friedkin and Johnsen's social influence network theory emphasizes that the interpersonal influence social structure (or social influence matrix) is the underling precondition for the group consensus or opinion convergence. In that model, the initial social influence structure of group of actors is assumed to be fixed during the entire process of opinion formation. However, with the evolution of time stamp, considering both stubborn and susceptible effects, the interpersonal influence structure can be regarded as a dynamic recursive process. For this reason, the interpersonal influence structure in their model is also dynamic, as described in Section 2.

From social influence matrix point of view, in large scale group opinions dynamic processes, the group belief is difficult to reach convergence, let alone consensus state, since social influence structures are generally unconnected, not to mention the social impact relations can be positive, negative or neutral. For example, on-line highlighted discussion, political or social hot spot debates often display polarization patterns [19].

In this paper, our interests concentrate on the precondition for consensus formation in a social group based on Friedkin's model. From interpersonal network structure point of view, our investigation presents the conditions for the

[1] *In classical sociological field, social control refers to the occurrence and effectiveness of ongoing efforts in a group to formulate, agree upon and implement collective courses of action.*

formation of group opinions convergence and consensus. We investigate the opinions convergence phenomenon over a group of N individuals with a random walk social influence structure, and for any given initial opinions distribution, i.e. the opinions evolution problem with a (time-variant) linear dynamic model driven by random matrices. Our analytic proof provides strict mathematic explanations for the deterministic characterization of the ergodicity, which can be used for studying the consensus over random graphs and the formation of opinion parties. The proof procedures are self-contained and based on ergodic theorem of Markov chain and eigenvalues of random graph, as introduced in Ref.[20].

2 Problem Formulation and Terminology

Social influence network theory presents a mathematical formalization of the social process of opinions changes that unfold in a social network of interpersonal influences. The spread of influence among individuals in a social network can be naturally modeled under a probabilistic framework, here, we briefly describe the classical Friedkin and Johnsen's model to illustrate how the opinion dynamics arise in the context of social networks.

Let $W = [w_{ij}]$ is a $N \times N$ row random matrix of interpersonal influence, i.e. for each i, w_{ij} denotes for the individual j's social influence to i, $\sum_j w_{ij} = 1$. $A = diag(a_1, a_2, ..., a_N)$ is a $N \times N$ diagonal matrix of individuals susceptibilities to interpersonal influence on the opinion. In a group of N persons, with the initial $N \times 1$ opinions vector $y^{(1)}$, the updating opinions vector $y^{(t)}$ in the interpersonal opinions influence system is described by Equ.(1),

$$y^{(t+1)} = AWy^{(t)} + (I - A)y^{(1)} \tag{1}$$

Definition 1. *The system (1) reaches the convergence state if, for any initial opinions vector $y^{(1)}$, it holds that $\lim_{t\to\infty} y^{(t)} = y^*$.*

Definition 2. *The system (1) reaches consensus state if, for any initial opinions vector $y^{(1)}$, and each $1 \leq i, j \leq N$, it holds that $\lim_{t\to\infty} |y_i^{(t)} - y_j^{(t)}| = 0$, where $|.|$ is the symbol of the absolute value. This means that, as a result of the social influence process, in the limit they have the same belief on the subject.*

As a consequence of system (1), the opinion profile at time $t \in Z \geq 0$ is equal to

$$y^{(t+1)} = \widehat{W}^t y^{(1)}, \tag{2}$$

where $\widehat{W}^t = (AW)^t + (\Sigma_{k=0}^{t-1}(AW)^k)(I - A)$ is the reduced-form coefficients matrix, discribing the total or net interpersonal effects that transform the initial opinions into equilibrium opinions, and for any entry $\widehat{w}_{ij}^t$ in $\widehat{W}^t$, satisfies $0 \leq \widehat{w}_{ij}^t \leq 1$, $\sum_j \widehat{w}_{ij}^t = 1$. According to Def.1, under suitable conditions, when

$t \to +\infty$ if $I - AW$ is nonsingular, the system (1) arrives at convergence equilibrium opinions profile y^*, where $y^* = \lim_{t\to\infty} y(t) = (I - AW)^{-1}(I - A)y^{(1)}$. When $t \to +\infty$, we have

$$\lim_{t\to\infty} \widehat{W}^t = \lim_{t\to\infty} \{(AW)^t + \sum_{k=0}^{t-1}(AW)^k(I - A)\} = (I - AW)^{-1}(I - A) = V. \quad (3)$$

Given large enough time stamp t, and a sufficiently small positive real number ε, V can be approximated by $\widehat{W}^t$. Furthermore, according to the approximation error $||\widehat{W}^t - V|| \leq \varepsilon$ (where $||.||$ denotes the matrix norm), we can obtain the time stamp's upper bound and lower bound as $ln(||V|| - \varepsilon)/ln(||\widehat{W}||) \leq t \leq ln(||V|| + \varepsilon)/ln(||\widehat{W}||)$, where $||\widehat{W}|| = ||AW + I - A||$.

Followed the same lines of the convergence results by Ishii and Tempo (2010) [21], and Golub and Jackson (2010) [22], by showing the ergodicity property, Frasca, et al.(2013) proved the convergence result of system (1)[18]. Touri and Nedic (2011) studied the ergodicity and consensus problem with a linear discrete-time dynamic model driven by stochastic matrices [23].

It should be noted according to Def.1, that equilibrium opinions may settle on the mean of group members' initial opinions, a compromise opinion that differs from the initial ones, or altered opinions that do not form a consensus. When consensus is formed in system (1), i.e. as $t \to +\infty$, $\widehat{W}^t$ will have the form of a stratification of individual contributions as following,

$$\widehat{W}^t = \begin{bmatrix} \widehat{w}_{11}^t & \widehat{w}_{22}^t & \dots & \widehat{w}_{NN}^t \\ \widehat{w}_{11}^t & \widehat{w}_{22}^t & \dots & \widehat{w}_{NN}^t \\ \vdots & \vdots & \vdots & \vdots \\ \widehat{w}_{11}^t & \widehat{w}_{22}^t & \dots & \widehat{w}_{NN}^t \end{bmatrix},$$

which suggests that the initial opinion of each individual makes a particular relative contribution to the emergent consensus.

3 Random Walk on Weighted Graph

In this section, without the lose of the generality of system (1), we firstly introduce the weighted adjacency random matrix, the weighted Laplacian and the transition matrix of the random walk, then we present the conditions for a group opinions consensus. Here we use the canonical graph symbol $G(V, E)$ in which V and E denote vertexes and edges respectively.

A weighted undirected graph G is defined as $w : V \times V \longrightarrow R$ such that $w_{ij} = w_{ji}$, if $\{i, j\} \notin E(G)$ then $w_{ij} = 0$. In the context, the weighted degree d_i of a vertex i is defined as $d_i = \sum_j w_{ij}$, $vol(G) = \sum_i d_i$ denotes the volume of the graph G. For a general weighted undirected graph G, the corresponding random walk is determined by transition probabilities $p_{ij} = Pr(x_{t+1} = j|x_t = i) = w_{ij}/d_i$, which are independent of i. Clearly, for each vertex i satisfies $0 \leq p_{ij} \leq 1, \sum_i p_{ij} = 1$, in other words, transition matrix P is row stochastic matrix. In addition if for any $j \in V(G)$ satisfying $\sum_j p_{ij} = 1$, then transition matrix P is named double stochastic matrix.

In this study, based on random walk on a graph, with the aim to prove the Friedkin and Johnsen's social influence system conclusions rigorously, we define transition matrix P on graph $\widehat{W^t}$ with entries $p_{ij} = Pr(x_{t+1} = j|x_t = i) = \widehat{w}^t_{ij}/\widehat{d}^t_i$, where $\widehat{d}^t_i = \sum_j \widehat{w}^t_{ij}$, and matrix L as follows:

$$L_{ij} = \begin{cases} \widehat{d}^t_i - \widehat{w}^t_{ii} & \text{if } i = j, \\ -\widehat{w}^t_{ij} & \text{if } i \text{ and } j \text{ are adjacent}, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

where $\widehat{w}^t_{ij} \in \widehat{W^t}$ is defined in Equations (2) and (3). Let T denote the diagonal matrix with the (i,i)-th entry having value $\widehat{d}^t_i$ as following

$$T = \begin{bmatrix} \widehat{d}^t_1 & \dots & \dots & 0 \\ 0 & \widehat{d}^t_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \widehat{d}^t_N \end{bmatrix}, \tag{5}$$

we set $T^{-1}(i,i) = 0$ for $\widehat{d}^t_i = 0$, and if $\widehat{d}^t_i = 0$ we say i is an isolated vertex. Then the graph $\widehat{W^t}$'s Laplacian matrix ζ is defined to be the form $\zeta = T^{-1/2} L T^{-1/2}$, and each entry in ζ is listed as following,

$$\zeta_{ij} = \begin{cases} 1 - \frac{\widehat{w}^t_{ii}}{\widehat{d}^t_i} & \text{if } i = j, \text{and } \widehat{d}^t_i \neq 0, \\ -\frac{\widehat{w}^t_{ij}}{\sqrt{\widehat{d}^t_i \widehat{d}^t_j}} & \text{if } i \text{ and } j \text{ are adjacent}, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Since ζ is symmetric, its eigenvalues are all real and non-negative. Let the eigenvalues of ζ be $\{\lambda_i | i = 0 : N-1\}$ in increasing order of λ_i, such that $0 = \lambda_0 \leq \lambda_1 \leq ... \leq \lambda_{N-1}$. Furthermore, it is easy to check that transition matrix P satisfies $P = T^{-1/2}(I - \zeta)T^{1/2}$, and $\mathbf{1}TP = \mathbf{1}T$, where $\mathbf{1}$ is unit vector.

Definition 3. *The random walk P^m is said to be irreducibility if for any $i, j \in V$, there exists some t such that $p^m_{ij} > 0$. Definition 3 ensures the graph P^m is strongly connected.*

Definition 4. *The random walk P^m is aperiodic if the greatest common divisor of the lengths of its simple cycles is 1, i.e., $gcd\{m : p^m_{ii} > 0\} = 1$ for any state i.*

Definition 5. *The random matrix P is said to be ergodic if there is an unique $n \times 1$ stationary distribution vector π satisfying $\lim_{m\to\infty} P^m (y^{(1)})' = \pi$, where $'$ is the transpose operation.*

Definition 6. *The random matrix P is convergent if $\lim_{m\to\infty} P^m (y^{(1)})'$ exists, for any initial vectors beliefs $y^{(1)}$.*

The social influence exchange among the N agents may be represented by a graph $G(V, E_m)$ with the set E_m of edges given by $E_m = \{(i,j)|p_{ij}^m > 0\}$. But this condition is not sufficient to guarantee consensus of dynamic system (1) as stated in Ref.[24]. This motivates the following stronger version Definition 7, as addressed in Refs.[25,26].

Definition 7. *(Bounded interconnectivity times). There is some $B \geq 1$ such that for each nodes pairs $(i,j) \in E_\infty$, agent j sends his/her social impact to neighbor i at least once at every B consecutive time slots, i.e. the graph $(G(P), E_m \bigcup ... \bigcup E_{(m+B-1)})$ is strongly connected. This condition is equivalent to the requirement that there exists $B \geq 1$ such that $(i,j) \in E_m \bigcup ... \bigcup E_{m+B-1}$ for all $(i,j) \in E_\infty$ and $m \geq 0$.*

Definition 5 is the well-known result that aperiodicity is necessary and sufficient for convergence in the case where P is strongly connected. In other words, the necessary conditions for the ergodicity of P are (i) *irreducibility*, (ii) *aperiodicity*, i.e., Def.5 is equivalent to Defs. 3 and 4. If Def.5 holds, Def.6 satisfies.

If a Markov chain is irreducible and aperiodic, i.e. Def.3 (or Def.3's stronger version Def.7) and Def.4 are both satisfied, or equivalently Def.5 holds, then P converges to its corresponding steady distribution. This conclusion is fairly easily verified by adapting theorems on steady-state distributions of Markov chains, such as the proof provided in Ref.[27]. From another alternative, we will prove this result by spectrum graph theorem in the following section.

For above Defs.3-7, we summarize the associated results in the following Theorem 1, then we emphasize on consensus result proof and converge time derivation.

Theorem 1. *If P is a random matrix, the following are equivalent:*
(i)P is aperiodic and irreducible.
(ii)P is ergodic.
(iii) P is convergent, there is a unique left eigenvector p_s of P corresponding to eigenvalue 1 whose entries sum to 1 such that, for every $y^{(1)}$,
$(\lim_{m\to\infty} P^m (y^{(1)})')_i = \pi(i), where \quad \pi(i) = (p_s)'(y^{(1)})'$ *for every i.*

Both (i) and (ii) in Theorem 1 are the well-known results. Next we focus on the proof of (iii) based on spectral graph theory. Theorem 1 presents the conditions for the formation of opinions convergence.

4 The Convergence of Opinions Profile on Random Graph

In this section, with the above Defs. 3,4 or 7, we prove that the convergence of group opinions over general weighted and undirected random graph are almost surely. In addition, we prove the lower bounds on the convergence time t for random walk P^t to be close to its stationary distribution, given an arbitrary initial distribution and small positive error ϵ. We note that this proof is based on spectrum graph theorem, which is different with Markov chains methods, such as in [9,10,11,18].

Proof. In a random walk associated with a weighted connected graph G, the transition matrix P satisfies $\mathbf{1}TP = \mathbf{1}T$, where $\mathbf{1}$ is the vector with all elements are scalar 1. Therefore the stationary distribution is exactly $\pi = \mathbf{1}T/vol(G)$. We show that for any initial opinions profile distribution $y^{(1)}$, when m is large enough, $P^m y^{(1)}$ converges to the stationary distribution π in the sense of L_2 or Euclidean norm. We write $y^{(1)}T^{-1/2} = \sum_i a_i e_i$, where e_i denotes the orthonormal eigenfunction associated with λ_i. Because $e_0 = \mathbf{1}T^{1/2}/\sqrt{vol(G)}$ and $< y^{(1)}, \mathbf{1} >= 1$, $||.||$ represents the L^2 norm, we have $a_0 = \frac{<y^{(1)}T^{-1/2}, \mathbf{1}T^{1/2}>}{||\mathbf{1}T^{1/2}||} = \frac{1}{\sqrt{vol(G)}}$. We then have

$$||y^{(1)}P^m - \pi|| = ||y^{(1)}P^m - \mathbf{1}T/vol(G)|| = ||y^{(1)}P^m - a_0 e_0 T^{1/2}||$$
$$= ||y^{(1)}T^{-1/2}(I-\zeta)^m T^{1/2} - a_0 e_0 T^{1/2}|| = ||\sum_{i\neq 0}(1-\lambda_i)^m a_i e_i T^{1/2}||$$
$$\leq (1-\lambda^{'})^m \frac{max_j\sqrt{\widehat{d_j^t}}}{min_j\sqrt{\widehat{d_j^t}}} \leq e^{-m\lambda^{'}} \frac{max_j\sqrt{\widehat{d_j^t}}}{min_j\sqrt{\widehat{d_j^t}}} \tag{7}$$

where

$$\lambda^{'} = \begin{cases} \lambda_1, & \text{if } 1-\lambda_1 \geq \lambda_{N-1} - 1 \\ 2-\lambda_{N-1}, & \text{else.} \end{cases}$$

Given any $\epsilon > 0$, for Equ.(7) we have

$$e^{-m\lambda^{'}} \frac{max_j\sqrt{\widehat{d_j^t}}}{min_j\sqrt{\widehat{d_j^t}}} \leq \epsilon, \tag{8}$$

then we have $\frac{max_j\sqrt{\widehat{d_j^t}}}{\epsilon \min_j \sqrt{\widehat{d_j^t}}} \leq e^{m\lambda^{'}}$, so $m \geq \frac{1}{\lambda^{'}} log(\frac{max_j\sqrt{\widehat{d_j^t}}}{\epsilon min_j\sqrt{\widehat{d_j^t}}})$.

With the symmetry of transition probability P^m, we easily check that $||y^{(1)}P^m - \pi^{'}|| = ||(y^{(1)}P^m - \pi^{'})^{'}|| = ||(y^{(1)}P^m)^{'} - \pi|| = ||(P^m)^{'}(y^{(1)})^{'} - \pi|| = ||P^m(y^{(1)})^{'} - \pi||$.

With this we conclude that after $m \geq [\frac{1}{\lambda'} log(\frac{max_j \sqrt{d_j}}{\epsilon min_j \sqrt{d_j}})]$ steps, the L_2 distance between $P^m(y^{(1)})^{'}$ and its stationary distribution $\pi^{'}$ is at most ϵ. Thus, P^m converges to a matrix with all of whose rows are equal to the positive vector $\pi^{'} = (\pi_1, \pi_2, ..., \pi_N)^{'}$, when a consensus is formed in Friedkin and Johnsen's model. Accordingly, we have $(\lim_{t \to \infty} y^{(t)})_i = \sum_{i=1}^{N} \pi_i y_i^{(1)}$ almost surely with ε approximating error corresponding to t updating steps.

In the herding example, there is consensus (of sorts), while which could lead to the wrong outcome or misunderstandings (misdirections) for the whole social group, such the "Mob phenomenon" of French revolution described by *Gustave LeBon*. In this case, group consensus is equivalent to the unwisdom of crowds. If group consensus to be emerged at certain slot t^*, such that $y^{(t^*)} = \frac{1}{N} \sum_{i=1}^{N} y_j^{(1)}$, for each j in a social group, we say that the society is wise, i.e. each individual arrives the group average initial opinions profile.

One special case of the above theorem is when P is a double random matrix. With this condition, the matrix has vector $\mathbf{1}$ as their common left eigenvector at all times, and therefore all the entries of the state vector converge to $(1/N)(\mathbf{1}^T y^{(1)})\mathbf{1} = (1/N) \sum_{j=1}^{N} y_j^{(1)} \mathbf{1}$, in other words, the mean of the initial N individual's opinion profile, with probability 1. This special case is addressed in Ref.[28], we say this group is a wise social group, as introduced in Ref.[22].

5 Conclusions

In this study, from random walk aspects, we investigate the well-known Friedkin and Johnsen's model. We define a weighted random walk P based on the social influence matrix. If P satisfies ergodicity, i.e. aperiodic and irreducible, Friedkin and Johnsen's model converges to the average consensus of the initial group opinions profile (the wise group decision making steady state) is almost surely. Furthermore, we prove the lower bounds on the convergence time m for random walk P^m to be close to its average consensus, given an arbitrary initial distribution and a small positive error ϵ.

Acknowledgments. This research was supported by National Natural Science Foundation of China under Grant Nos.71171187 and 61473284, and the Scientific Research Foundation of Yunnan Province.

References

1. DeGroot, M.: Reaching a consensus. Journal of the American Statistical Association **69**(345), 118–121 (1974)
2. Friedkin, N.E., Johnsen, E.C.: Social influence networks and opinion change. In: Lawler, E.J., Macy, M.W. (eds.) Advances in Group Processes vol. 16, pp. 1–29 (1999)
3. Friedkin, N.E.: A formal theory of reflected appraisals in the evolution of power. Administrative Science Quarterly **56**(4), 501–529 (2011)

4. French, J.: A formal theory of social power. Psychological Review **63**, 181–194 (1956)
5. Jadbabaie, A., Lin, J., Morse, A.S.: Coordination of groups of mobile autonomous agents using nearest neighbor rules. IEEE Transactions on Automatic Control **48**(6), 988–1001 (2003)
6. Blondel, V., Hendrickx, J.M., Olshevsky, A., et al.: Convergence in multiagent coordination, consensus, and flocking. In: The 44th IEEE Conference on Decision and Control and the European Control Conference 2005, pp. 2996–3000 (2005)
7. Olfati-Saber, R., Murray, R.M.: Consensus problems in networks of agents with switching topology and time-delays. IEEE Transactions on Automatic Control **49**(9), 1520–1533 (2004)
8. Moreau, L.: Stability of multiagent systems with time-dependent communication links. IEEE Transactions on Automatic Control **50**(2), 169–182 (2005)
9. Fagnani, F., Zampieri, S.: Randomized consensus algorithms over large scale networks. IEEE Journal on Selected Areas in Communications **26**(4), 634–649 (2008)
10. Tsitsiklis, J.N.: Problems in decentralized decision making and computation. Ph.D. Dissertation, Massachusetts Institute of Technology (1985)
11. Kempe, D., Dobra, A., Gehrke, J.: Gossip-based computation of aggregate information. In: Proceedings of the 44th Annual IEEE Symposium on Foundations of Computer Science, pp. 482–491. IEEE (2003)
12. Intanagonwiwat, C., Govindan, R., Estrin, D.: Directed diffusion: a scalable and robust communication paradigm for sensor networks. In: Proceedings of the 6th Annual International Conference on Mobile Computing and Networking, pp. 56–67. ACM (2000)
13. Zhao, J., Govindan, R., Estrin, D.: Computing aggregates for monitoring wireless sensor networks. In: Proceedings of the First IEEE International Workshop on Sensor Network Protocols and Applications, pp. 139–148. IEEE (2003)
14. Xiao, L., Boyd, S., Lall, S.: A scheme for robust distributed sensor fusion based on average consensus. In: The 4th International Symposium on Information Processing in Sensor Networks, 63–70. IEEE (2005)
15. Dimakis, A.G., Sarwate, A.D., Wainwright, M.J.: Geographic gossip: efficient aggregation for sensor networks. In: Proceedings of the 5th International Conference on Information Processing in Sensor Networks, pp. 69–76. ACM (2006)
16. Galam, S.: From 2000 bush-gore to 2006 italian elections: voting at fifty-fifty and the contrarian effect. Quality and Quantity Journal **41**, 579–589 (2007)
17. Arrow, K.J.: Social Choice and Individual Values. Wiley, New York (1951)
18. Frasca, P., Ravazzi, C., Tempo, R., et al.: Gossips and prejudices: Ergodic randomized dynamics in social networks. arXiv:1304.2268 (2013)
19. Yardi, S., Boyd, D.: Dynamic Debates: An Analysis of Group Polarization Over Time on Twitter. Bulletin of Science, Technology & Society **30**(5), 316–327 (2010)
20. Chung, F.R.K.: Spectral graph theory. American Mathematical Soc. 92 (1997)
21. Ishii, H., Tempo, R.: Distributed randomized algorithms for the PageRank computation. IEEE Transactions on Automatic Control **55**(9), 1987–2002 (2010)
22. Golub, B., Jackson, M.O.: Naive learning in social networks and the wisdom of crowds. American Economic Journal: Microeconomics, 112–149 (2010)
23. Touri, B., Nedic, A.: On ergodicity, infinite flow, and consensus in random models. IEEE Transactions on Automatic Control **56**(7), 1593–1605 (2011)
24. Bertsekas, D., Tsitsiklis, J.N.: Parallel and Distributed Computation: Numerical Methods. Prentice Hall (1989)

25. Nedic, A., Ozdaglar, A.: Distributed subgradient methods for multi-agent optimization. IEEE Transactions on Automatic Control **54**(1), 48–61 (2009)
26. Olshevsky, A., John, N.: Tsitsiklis. Convergence Speed in Distributed Consensus and Averaging. SIAM Journal on Control and Optimization **48**(1), 33–55 (2009)
27. Kemeny, J.G., Snell, J.L.: Finite Markov Chains. Springer (1960)
28. Boyd, S., et al.: Randomized gossip algorithms. IEEE Trans. Information Theory **52**(6), 2508–2530 (2006)