Hesitant Fuzzy Information Aggregation With A Prioritization Relationship Between Attributes

$JiangangFeng^1$
1.Management School and
Qufu Normal University
China, Rizhao 276826
tysyfjg@163.com

$CuipingWei^2$
2.College of Mathematical Science
Yangzhou University
China, Jiangsu Yangzhou 225002
wei_cuiping@aliyun.com

$XijinTang^3$ and
$ZhiliangRen^1$
3.Academy of Mathematics and Systems Science
Chinese Academy of Sciences
China, Beijing 100190
xjtang@iss.ac.cn

***Abstract*—This paper proposes a method to determine weight vector of attributes with linear prioritization relationship and the assessment values are represented by hesitant fuzzy elements. The weight associated with an attribute depends upon the satisfaction of an alternative for the higher priority attribute. First, some hesitant fuzzy prioritized aggregation operators are defined and their desirable properties are discussed. These proposed operators can capture the prioritization phenomenon among the aggregated hesitant fuzzy elements. Then, we develop a multi-attribute decision-making (MADM) method based on the proposed operators in hesitant fuzzy environment. Finally, a practical example is provided to demonstrate the effectiveness of the developed method and make a comparative analysis on generalized hesitant fuzzy prioritized aggregation operators in decision-making.**

***Keywords*—Multi-attribute decision-making, Hesitant fuzzy sets, Prioritized aggregation operators**

I. Introduction

Torra and Narukawa [1] and Torra [2] defined the hesitant fuzzy set which allows the membership to have a set of possible values, some basic operations. Then, they studied its relationship with intuitionistic fuzzy set and fuzzy multisets. Afterwards, in order to aggregate the hesitant fuzzy information, Xia and Xu [3] proposed a series of aggregation operators under various situations and discussed the relationship among them. Then, they applied the developed aggregation operators to solve group decision-making problems with anonymity. Under the assumption that the values in all hesitant fuzzy elements are arranged in an increasing order and two hesitant fuzzy elements are of same length for comparison, Xu et al. [4], [5] and Chen et al. [6] defined a variety of distance measures, similarity measures and correlation measures, and then discussed their properties in detail. In addition, they proposed a number of hesitant ordered weighted distance measures and hesitant ordered weighted similarity measures. Xu et al. [7] developed some aggregation operators for hesitant fuzzy elements with the aid of quasi-arithmetic means, and gave two methods of determining the weight vectors with the aggregation operators based on the support degrees between the aggregated arguments and Choquet integral. Then, they proposed the corresponding decision-making method. Gu et al. [8] proposed the evaluation decision-making method for risk investment with hesitant fuzzy information based on the hesitant fuzzy weighted averaging (HFWA) operator, and demonstrated its effectiveness by an illustrative example. Xu and Xia [9] introduced the concepts of entropy and cross-entropy for hesitant fuzzy information, discussed their properties, and developed several measures formulas of entropy and cross-entropy. Then, they analyzed the relationship between them and similarity measure. Finally, they proposed two MADM methods based on the TPOSIS method.

From those results, we can know that hesitant fuzzy set is a very useful tool to deal with uncertainty and some MADM theories and methods have been developed under the hesitant fuzzy environment. However, above proposed MADM methods for HFEs are under the assumption that the attributes are at the same priority level. They are characterized by the ability to trade off between attributes. For example, if G_i and G_k are two attributes with weight ω_i and ω_k respectively, in the decision-making method developed above, we can compensate for a decrease of Δ in satisfaction to attribute G_i by gain $\frac{\omega_i}{\omega_k}\Delta$ in satisfaction to attribute G_k. However, the attributes have different priority level in many real decision-making problems, so this kind of compensation between attributes is not feasible. A typical example is in the case of buying a car upon two attributes safety and cost. We give the assumption that attribute safety has a higher priority than attribute cost, it indicates that we are not wiling to trade off satisfaction of attribute cost until perhaps we attain some level of satisfaction of attribute safety. Using the weighted aggregation operators to model the prioritized MADM is a effective method. Yager [10] showed that the prioritization of attributes can be modeled by using importance weights in which the weights associated with the lower priority attribute are related to the satisfaction of the higher priority attribute. To develop this concept, Yager [11] further proposed a prioritized averaging/scoring aggregation operator with a strict/weak priority order by means of the product t-norm, and the prioritized “and” operator and the prioritized “or” operator. Yager [12] proposed the prioritized OWA operator. Furthermore, taking DM’s requirements into account, Chen and Wang [13] and Wang and Chen [14] found the drawbacks of the method presented in Ref [10] by some numerical examples and suggested that the weights of the lower priority attribute depend on whether each alternative satisfies the requirements of all the higher priority attribute or not, proposed a generalized prioritized MADM method which overcome the drawbacks. Although previous researches have greatly developed the priority weighted MADM, there were still some limitations and drawbacks. Yan et al. [15] proposed aggregation operators to overcome the limitations of previous works, and showed the effectiveness and advantages of the proposed approach by comparative analysis with Ref [13], [14]. Wei and Tang [16] proposed generalized prioritized aggregation operators based on the WOWA operator. In intuitionistic fuzzy environment, Yu and Xu [17] proposed the intuitionistic fuzzy prioritized aggregation operator, gave a determining weighted method (IF-BUM) and developed the intuitionistic fuzzy prioritized OWA operator. Yu et al. [18] proposed the aggregation method for IVIFVs which has prioritization relationship between the aggregated arguments. Motivated by the ideal of prioritized aggregation operators on the condition of the linear ordered attributes, Wei [19] proposed the hesitant fuzzy priority weighted average (HF-PWA) operator for hesitant fuzzy information, and developed corresponding approaches to solve the hesitant fuzzy MADM problems, in which the attributes are at different priority levels.

In this paper, we continue the research on the aggregation method for HFEs which has prioritized relationship between the aggregated arguments. The reminder of this paper is organized as follows. In section 2, we briefly review some basic knowledge. In section 3, we first propose a method for determining weight vector of the attributes for a linear order. The weight associated with an attribute depends upon the satisfaction of the higher priority attributes by modeling the prioritization between attributes. Then, based on it and Ref [19], we define the hesitant fuzzy prioritized weighted averaging (HFPWA) operator. In section 4, we define several generalized hesitant fuzzy prioritized aggregation operators and discuss their desirable properties. In section 5, we develop a method for MADM based on proposed operators for hesitant fuzzy environment. Section 6 gives an practical example and makes a comparative analysis on generalized hesitant fuzzy prioritized aggregation operators in decision-making.

II. Preliminaries

Definition 2.1: [1], [2] Let X be a reference set, a hesitant fuzzy set (HFS) on X is in terms of a function that when applied to X returns a subset of [0,1], which can be represented as the following mathematical symbol: $E = <x, h_E(x)|x \in X>$, where $h_E(x)$ is a set of some value in [0,1], denoting the possible membership degrees of the elements $x \in X$ to the set E. For convenience, Xia and Xu [3] call $h_E(x)$ a hesitant fuzzy element (HFE) and H the set of all the HEEs.

For three HFEs $h, h_1, h_2 \in H$, $\lambda \in R$, Torra [2], Xia and Xu [3] gave some operations on them, shown as:

(1) $h^c = \bigcup_{\gamma\in h} \{1-\gamma\}$,

(2) $h_1 \cup h_2 = \bigcup_{\gamma_1\in h_1, \gamma_2\in h_2} \{max(\gamma_1,\gamma_2)\}$,

(3) $h_1 \cap h_2 = \bigcup_{\gamma_1\in h_1, \gamma_2\in h_2} \{min(\gamma_1,\gamma_2)\}$,

(4) $h^\lambda = \bigcup_{\gamma\in h} \{\gamma^\lambda\}$,

(5) $\lambda h = \bigcup_{\gamma\in h} \{1-(1-\gamma)^\lambda\}$,

(6) $h_1 \oplus h_2 = \bigcup_{\gamma_1\in h_1, \gamma_2\in h_2} \{\gamma_1+\gamma_2-\gamma_1\gamma_2\}$,

(7) $h_1 \otimes h_2 = \bigcup_{\gamma_1\in h_1, \gamma_2\in h_2} \{\gamma_1\gamma_2\}$.

Torra and Narukawa [1] proposed an extended principle on HFEs.

Definition 2.2: [1] Let Θ be a function $\Theta : [0,1]^n \to [0,1]$ and let H be a set of n hesitant fuzzy sets on the refenrece set X($i.e., H = \{h_1, h_2, \cdots, h_n\}$ are hesitant fuzzy sets on X). Then, the extention of Θ on H is defined for each x in X by:

$$\Theta_H(h_1, h_2, \cdots, h_n) = \bigcup_{\gamma\in\{h_1\times h_2\times\cdots\times h_n\}} \{\Theta(\gamma)\}. \quad (2.1)$$

Definition 2.3: [3] For a HFE h, $s(h) = \frac{1}{|h|}\sum_{\gamma\in h}\gamma$ is called the score function of h, let $|h|$ be the number of values in h. For two HFEs h_1 and h_2, if $s(h_1) > s(h_2)$, then h_1 is superior to h_2, denoted by $h_1 \succ h_2$; if $s(h_1) = s(h_2)$, then h_1 is indifferent to h_2, denoted by $h_1 \sim h_2$.

Based on Definition 2.1 and the defined operations for HFEs, Xu and Xia [3] gave the hesitant fuzzy weighted averaging (HFWA) operator [3]:

$$\begin{aligned} HFWA(h_1, h_2, \cdots, h_n) &= \bigoplus_{j=1}^{n}(\omega_j h_j) = \\ &= \bigcup_{\gamma_1\in h_1,\cdots,\gamma_n\in h_n} \{1-\prod_{j=1}^{n}(1-\gamma_j)^{\omega_j}\}, \end{aligned} \quad (2.2)$$

where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$ is the weight vector of $h_j (j = 1, 2, \cdots, n)$ with $\omega_j \in [0,1]$ and $\sum_{j=1}^{n}\omega_j = 1$.

It is noted that they are satisfying Idempotency, Monotonicity and Boundedness.

III. Hesitant Fuzzy Prioritized Aggregation Operator and Their Properties

A. The HFPWA operator

Under hesitant fuzzy environment, suppose that $G = (G_1, G_2, \cdots, G_n)$ be a collection of attributes and there is a prioritized relation between the attributes expressed by the linear ordering $G_1 \succ G_2 \succ \cdots \succ G_n$, indicate attribute G_j has a higher priority to G_k, if $j < k$. In this case, the weight associated with an attribute depends upon the satisfaction of the higher priority attributes by modeling the prioritization between attributes, let h_j be the hesitant fuzzy evaluation value under the attribute $G_j (j = 1, 2, \cdots, n)$ for some alternative. According to [10], [13], [15], we defined:

$$\begin{aligned} T_1 = \{1\}, T_j = T_{j-1} \wedge h_j, j = 2, 3, \cdots, n; \\ l_1 = 1, l_j = \prod_{k=1}^{j} s(T_j), j = 2, 3, \cdots, n, \end{aligned} \quad (3.3)$$

then we can calculate the weight of the hesitant fuzzy evaluation value $h_j (j = 1, 2, \cdots, n)$ by the mean of l_i:

$$\omega_j = \frac{l_j}{\sum_{i=1}^{n} l_i}, j = 1, 2, \cdots, n. \quad (3.4)$$

Based on this weight-determined technics and Ref [19], we defined hesitant fuzzy prioritized weight averaging operator.

Definition 3.1: Let $h_j(j=1,2,\cdots,n)$ be a collection of HFEs, and $\omega=(\omega_1,\omega_2,\cdots,\omega_n)$ be the weight vector of $h_j(j=1,2,\cdots,n)$, such that $\omega_j=\frac{l_j}{\sum_{i=1}^{n} l_i}$, $T_1=\{1\}, T_j=T_{j-1}\bigwedge h_j(j=2,\cdots,n)$, $l_1=1, l_j=\prod_{i=1}^{j} s(T_i)(j=2,\cdots,n)$, and $s(T_j)$ is the score value of $T_j(j=1,2,\cdots,n)$. Then we define the hesitant fuzzy prioritized weighted average (HFPWA) operator as follows:

$$\begin{aligned} HFPWA(h_1,h_2,\cdots,h_n)=&\frac{l_1}{\sum_{i=1}^{n} l_i}h_1\oplus\frac{l_2}{\sum_{i=1}^{n} l_i}h_2\oplus \\ \cdots\oplus\frac{l_n}{\sum_{i=1}^{n} l_i}h_n=&\bigoplus_{j=1}^{n}\left(\frac{l_j}{\sum_{i=1}^{n} l_i}h_j\right). \end{aligned} \tag{3.5}$$

Analogous to Ref [19], it can be easily proved that HFPWA operator has the Idempotency, Monotonicity and Boundedness.

Theorem 3.1: Let $h_j(j=1,2,\cdots,n)$ be a collection of HFEs, and $\omega=(\omega_1,\omega_2,\cdots,\omega_n)$ be the weight vector of $h_j(j=1,2,\cdots,n)$, such that $\omega_j=\frac{l_j}{\sum_{i=1}^{n} l_i}$,$T_1=\{1\}, T_j=T_{j-1}\bigwedge h_j(j=2,\cdots,n)$, $l_1=1, l_j=\prod_{i=1}^{j} s(T_i)(j=2,\cdots,n)$, and $s(T_j)$ is the score value of $T_j(j=1,2,\cdots,n)$. Then their aggregated value by using the HFPWA operator is also a HFE, and

$$\begin{aligned} &HFPWA(h_1,h_2,\cdots,h_n)= \\ &\frac{l_1}{\sum_{i=1}^{n} l_i}h_1\oplus\frac{l_2}{\sum_{i=1}^{n} l_i}h_2\oplus\cdots\oplus\frac{l_n}{\sum_{i=1}^{n} l_i}h_n= \\ &\bigoplus_{j=1}^{n}\left(\frac{l_j}{\sum_{i=1}^{n} l_i}h_j\right)= \\ &\bigcup_{\gamma_1\in h_1,\cdots,\gamma_n\in h_n}\left\{1-\prod_{j=1}^{n}(1-\gamma_j)^{\frac{l_j}{\sum_{i=1}^{n} l_i}}\right\}. \end{aligned} \tag{3.6}$$

B. Generalized hesitant fuzzy prioritized aggregation operators

Based on the definitions and the properties of the HFPWA operator in subsection A, we propose some generalized hesitant fuzzy prioritized aggregation operators, and give some properties.

Definition 3.2: Let $h_j(j=1,2,\cdots,n)$ be a collection of HFEs, $\lambda>0$, and $\omega=(\omega_1,\omega_2,\cdots,\omega_n)$ be the weight vector of $h_j(j=1,2,\cdots,n)$, such that $\omega_j=\frac{l_j}{\sum_{i=1}^{n} l_i}$,$T_1=1, T_j=T_{j-1}\bigwedge h_j(j=2,\cdots,n)$, $l_1=1, l_j=\prod_{i=1}^{j} s(T_i)(j=2,\cdots,n)$, and $s(T_j)$ is the score value of $T_j(j=1,2,\cdots,n)$. If

$$\begin{aligned} &GHFPWA_\lambda(h_1,h_2,\cdots,h_n)= \\ &\left(\frac{l_1}{\sum_{i=1}^{n} l_i}(h_1)^\lambda\oplus\frac{l_2}{\sum_{i=1}^{n} l_i}(h_2)^\lambda\oplus\cdots\oplus\frac{l_n}{\sum_{i=1}^{n} l_i}(h_n)^\lambda\right)^{\frac{1}{\lambda}} \\ &=\left(\bigoplus_{j=1}^{n}\left(\frac{l_j}{\sum_{i=1}^{n} l_i}(h_j)^\lambda\right)\right)^{\frac{1}{\lambda}}= \\ &\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{\left(1-\prod_{j=1}^{n}(1-\gamma_j^\lambda)^{\frac{l_j}{\sum_{i=1}^{n} l_i}}\right)^{\frac{1}{\lambda}}\right\}, \end{aligned} \tag{3.7}$$

then $GHFPWA$ is called a generalized hesitant fuzzy prioritized weighted average operator.

If $\lambda=1$, then the $GHFPWA$ operator becomes the $HFPWA$ operator.

According to Ref [19], It can be proved that the $GHFPWA$ operator has followed the properties:

(1) Idempotency: If all HFE $h_j(j=1,2,\cdots,n)$ are equal, i.e. $h_j=h$, for all j, then $GHFPWA_\lambda(h_1,h_2,\cdots,h_n)=h$.

(2) Monotonicity: Let $h_j(j=1,2,\cdots,n)$ be a collection of HFEs, $h_j^{'}(j=1,2,\cdots,n)$ is also a collection of HFEs, and $h_j\le h_j^{'}$, then $GHFPWA_\lambda(h_1,h_2,\cdots,h_n)\le GHFPWA_\lambda(h_1^{'}, h_2^{'},\cdots,h_n^{'})$.

(3) Boundedness: Let $h_j(j=1,2,\cdots,n)$ be a collection of HFEs, and $h_-=\min_j h_j$, $h_+=\max_j h_j$, then $h_-\le GHFPWA_\lambda(h_1,h_2,\cdots,h_n)\le h_+$.

Lemma 3.1: [20][21] Let $x_j>0,\lambda_j, j=1,2,\cdots,n$, and $\sum_{j=1}^{n}\lambda_j=1$, then $\prod_{j=1}^{n}x_j^{\lambda_j}\le\sum_{j=1}^{n}\lambda_j x_j$ with equality if only if $x_1=x_2=\cdots=x_n$.

Theorem 3.2: Let $h_j(j=1,2,\cdots,n)$ be a collection of HFEs, and $\lambda>0$, $\omega=(\omega_1,\omega_2,\cdots,\omega_n)$ the weight vector of $h_j(j=1,2,\cdots,n)$, such that $\omega_j=\frac{l_j}{\sum_{i=1}^{n} l_i}$,$T_1=1, T_j=T_{j-1}\bigwedge h_j(j=2,\cdots,n)$, $l_1=1, l_j=\prod_{i=1}^{j} s(T_i)(j=2,\cdots,n)$, and $s(T_j)$ is the score value of $T_j(j=1,2,\cdots,n)$.

Based on the hesitant fuzzy prioritized operator and the quasi-arithmetic means [19], we can get the quasi hesitant fuzzy prioritized operator, shown as:

Definition 3.3: Let $h_j(j=1,2,\cdots,n)$ be a collection of HFEs, and $\omega=(\omega_1,\omega_2,\cdots,\omega_n)$ the weight vector of $h_j(j=1,2,\cdots,n)$, such that $\omega_j=\frac{l_j}{\sum_{i=1}^{n} l_i}$,$T_1=1, T_j=T_{j-1}\bigwedge h_j(j=2,\cdots,n)$, $l_1=1, l_j=\prod_{i=1}^{j} s(T_i)(j=2,\cdots,n)$, and $s(T_j)$ is the score value of $T_j(j=1,2,\cdots,n)$.

If

$$QHFPWA(h_1,h_2,\cdots,h_n)=$$
$$f^{-1}\left(\frac{l_1}{\sum\limits_{i=1}^{n}l_i}f(h_1)\oplus\frac{l_2}{\sum\limits_{i=1}^{n}l_i}f(h_2)\oplus\cdots\oplus\frac{l_n}{\sum\limits_{i=1}^{n}l_i}f(h_n)\right)$$
$$=f^{-1}\left(\bigoplus_{j=1}^{n}\left(\frac{l_j}{\sum\limits_{i=1}^{n}l_i}f(h_j)\right)\right)= \tag{3.8}$$
$$\bigcup_{\gamma_1\in h_1,\cdots,\gamma_n\in h_n}\left\{f^{-1}\left(1-\prod_{j=1}^{n}(1-f(\gamma_j))^{\frac{l_j}{\sum_{i=1}^{n}l_i}}\right)\right\},$$

then $QHFPWA$ is called a quasi hesitant fuzzy prioritized weighted average operator, where $f:[0,1]\to[0,1]$ is a strictly continuous monotonic function.

If $f(x)=x$, then the $QHFPWA$ operator becomes the $HFPWA$ operator.

If $f(x)=x^{\lambda}$, $\lambda>0$, then the $QHFPWA$ operator becomes the $GHFPWA$ operator.

In fact, based on the ordered modular averages (OMAs) [22], we can further generalize the hesitant fuzzy prioritized operator as follows:

Definition 3.4: Let $h_j(j=1,2,\cdots,n)$ be a collection of HFEs, and $\omega=(\omega_1,\omega_2,\cdots,\omega_n)$ the weight vector of $h_j(j=1,2,\cdots,n)$, such that $\omega_j=\frac{l_j}{\sum_{i=1}^{n}l_i}$,$T_1=1,T_j=T_{j-1}\bigwedge h_j(j=2,\cdots,n)$, $l_1=1,l_j=\prod_{i=1}^{j}s(T_i)(j=2,\cdots,n)$, and $s(T_j)$ is the score value of $T_j(j=1,2,\cdots,n)$. If

$$HFPMWA(h_1,h_2,\cdots,h_n)=$$
$$\frac{l_1}{\sum\limits_{i=1}^{n}l_i}f_1(h_1)\oplus\frac{l_2}{\sum\limits_{i=1}^{n}l_i}f_2(h_2)\oplus\cdots\oplus\frac{l_n}{\sum\limits_{i=1}^{n}l_i}f_n(h_n)$$
$$=\bigoplus_{j=1}^{n}\left(\frac{l_j}{\sum\limits_{i=1}^{n}l_i}f_j(h_j)\right)=$$
$$\bigcup_{\gamma_1\in h_1,\gamma_2\in h_2,\cdots,\gamma_n\in h_n}\left\{1-\prod_{j=1}^{n}(1-f(\gamma_j))^{\frac{l_j}{\sum_{i=1}^{n}l_i}}\right\},$$

then $HFPMWA$ is called a hesitant fuzzy prioritized modular weighted average operator, where $f_i:[0,1]\to[0,1](i=1,2\cdots,n)$ is a strictly continuous monotonic function.

If $f_i(x)=x(i=1,2\cdots,n)$, then the $HFPMWA$ operator becomes the $HFPWA$ operator.

IV. An Approach to Multi-Attribute Decision-making Under Hesitant Fuzzy Environment

In this section, we utilize the proposed hesitant fuzzy prioritized aggregation operators to solve group decision-making problems under hesitant fuzzy environment. In a group decision-making problem, suppose $X=\{x_1,x_2,\cdots,x_m\}$ is the set of alternatives, let $G=(G_1,G_2,\cdots,G_n)$ be a collection of attributes and there is a prioritized relation between these attributes expressed by the linear ordering $G_1\succ G_2\succ\cdots\succ G_n$, indicate attribute G_j has a higher priority G_k, if $j<k$. If decision makers provide all the possible evaluated values under the attribute G_j for the alternative x_i with anonymity, these values can be considered as a hesitant fuzzy element h_{ij}. In the case where two decision makers provide the same value, then the value emerges only once in h_{ij}. Suppose that the decision matrix $H=(h_{ij})_{m\times n}$ is the hesitant fuzzy decision matrix, where $h_{ij}(i=1,2,\cdots,m,j=1,2,\cdots,n)$ is in the form of HFEs. Then, based on the generalized hesitant fuzzy prioritized aggregation (GHFPWA and QHFPWA) operators, we give a method for group decision-making with hesitant fuzzy information, which involves the following steps:

Step 1. Calculate the weights ω_{ij} of $h_{ij}(i=1,2,\cdots,m,j=1,2,\cdots,n)$, as follow:

$$\begin{gathered}T_{i1}=\{1\},i=1,2,\cdots,m,\\ T_{ij}=T_{i,j-1}\wedge h_{ij}(i=1,2,\cdots,m,j=2,\cdots,n)\end{gathered} \tag{4.9}$$

$$\begin{gathered}l_{i1}=1,i=1,2,\cdots,m,\\ l_{ij}=\prod_{k=1}^{j}s(T_{ij})(i=1,2,\cdots,m,j=2,\cdots,n),\end{gathered} \tag{4.10}$$

$$\omega_{ij}=\frac{l_{ij}}{\sum\limits_{j=1}^{n}l_{ij}}(i=1,2,\cdots,m,j=1,2,\cdots,n). \tag{4.11}$$

Step 2. Aggregate the hesitant fuzzy values $h_{ij}(j=1,2,\cdots,n)$ by using the hesitant fuzzy prioritized extension, denoted by Θ, then

$$h_i=\Theta(h_{i1},h_{i2},\cdots,h_{in}),i=1,2,\cdots,m.$$

Θ can be someone of the GHFPWA operator and the QHFPWA operator.

Step 3. Calculate the scores $s(h_i)(i=1,2,\cdots,m)$ of the overall hesitant fuzzy preference values $h_i(i=1,2,\cdots,m)$ and rank them.

Step 4. Rank all the alternatives $A_i(i=1,2,\cdots,m)$ and select the best one(s) accordance with $s(h_i)(i=1,2,\cdots,m)$,

Step 5. End.

V. Practical Example

Working to strengthen academic education and promoting the building of teaching body, the school of management in a Chinese university wants to recruit oversea outstanding faculties. This program has been raised great attention. University president e_1, dean of management school e_2, and human resource officer e_3 sets up the panel of recruitment to take the whole responsibility for this program. They have made strict evaluation for 5 candidates $x_i(i=1,2,3,4,5)$ from four aspects, namely morality G_1, research capability G_2, teaching skill G_3, education background G_4. In addition, this program is in strict accordance with the principle of combine ability with political integrity. The prioritization relationship for attributes is shown as: $G_1\succ G_2\succ G_3\succ G_4$, The five candidates $x_i(i=1,2,3,4,5)$ are to be evaluated by the three decision makers under the above four attributes with anonymity, and construct the hesitant decision matrix $H=(h_{ij})_{5\times 4}$, which is shown in Table 1.

$Table1 : Hesitant\ fuzzy\ decision\ matrix\ H$

	G_1	G_2	G_3	G_4
x_1	$\{0.4, 0.5, 0.7\}$	$\{0.5, 0.8\}$	$\{0.6, 0.7, 0.9\}$	$\{0.5, 0.6\}$
x_2	$\{0.6, 0.7, 0.8\}$	$\{0.5, 0.6\}$	$\{0.4, 0.6, 0.7\}$	$\{0.4, 0.5\}$
x_3	$\{0.6, 0.8\}$	$\{0.2, 0.3, 0.5\}$	$\{0.4, 0.6\}$	$\{0.5, 0.7\}$
x_4	$\{0.5, 0.6, 0.7\}$	$\{0.4, 0.5\}$	$\{0.8, 0.9\}$	$\{0.3, 0.4, 0.5\}$
x_5	$\{0.6, 0.7\}$	$\{0.5, 0.7\}$	$\{0.7, 0.8\}$	$\{0.2, 0.3, 0.4\}$

Then, in order to get the optimal result, we make an example of the GHFPWA operator($\lambda = 1$) to develop an approach to MADM problem under hesitant fuzzy information, the main step is described as following:

Step 1. Utilize (4.10) and (4.11) to calculate the values of $\omega_{ij}(i = 1, 2, \cdots, m, j = 1, 2, \cdots, n)$ as follow:

$$\omega_{ij} = \begin{bmatrix} 0.4339 & 0.2820 & 0.1833 & 0.1008 \\ 0.5132 & 0.2822 & 0.1411 & 0.0635 \\ 0.6734 & 0.2245 & 0.0783 & 0.0274 \\ 0.5769 & 0.2596 & 0.1168 & 0.0467 \\ 0.4836 & 0.2901 & 0.1741 & 0.0522 \end{bmatrix}$$

Step 2. Aggregate all hesitant fuzzy values $h_{ij}(j = 1, 2, \cdots, n)$ by using the (GHFPWA) operator to derive the overall hesitant fuzzy values $h_i(i = 1, 2, 3, 4, 5)$ of the candidates x_i. If $\lambda = 1$, take alternative x_1 for an example, we have

$h_1 = GHFPWA_\lambda(h_{11}, h_{12}, h_{13}, h_{14}) =$
$GHFPWA_\lambda(\{0.4, 0.5, 0.7\}, \{0.5, 0.8\}, \{0.6, 0.7, 0.9\},$
$\{0.5, 0.6\}) =$

$$\bigcup_{\gamma_{11}\in h_{11},\gamma_{12}\in h_{12},\gamma_{13}\in h_{13},\gamma_{14}\in h_{14}} \left\{1 - \prod_{j=1}^{4} (1 - \gamma_{1j})^{\omega_{1j}}\right\}$$

$= \{0.4805, 0.4920, 0.5072, 0.5181, 0.5971, 0.6060, 0.5988,$
$0.6077, 0.6194, 0.6278, 0.6888, 0.6957, 0.5200, 0.5307,$
$0.5446, 0.5548, 0.6277, 0.6360, 0.6293, 0.6375, 0.6483,$
$0.6561, 0.7125, 0.7188, 0.6154, 0.6240, 0.6352, 0.6433,$
$0.7017, 0.7083, 0.7030, 0.7096, 0.7182, 0.7245, 0.7696,$
$0.7747\}$

Step 3. Calculate the scores $s(h_i)(i = 1, 2, 3, 4, 5)$ of the overall hesitant fuzzy values $h_i(i = 1, 2, 3, 4, 5)$ of the candidates x_i: If $\lambda = 1$, $s(h_1) = 0.6329, s(h_2) = 0.6376, s(h_3) = 0.6306, s(h_4) = 0.6100, s(h_5) = 0.6486$, then $s(h_5) > s(h_2) > s(h_1) > s(h_3) > s(h_4)$.

Step 4. Rank all the candidates $x_i(i = 1, 2, 3, 4, 5)$ in accordance with the scores $s(h_i)(i = 1, 2, 3, 4, 5)$ of the overall hesitant fuzzy values $h_i(i = 1, 2, 3, 4, 5)$: $x_5 \succ x_2 \succ x_1 \succ x_3 \succ x_4$ and thus the most desirable candidate is x_5.

If λ get different values, we can make further analysis on the sequence of alternative based on the GHFPWA operator, followed in Table 2.

From Table 2, we find that the larger λ the greater the score value of the overall hesitant fuzzy values for alternatives; when λ is given different values, the effect of the aggregated results are different in the sequence of alternatives, so the sequence for alternatives is different. To analyze the sequence of alternatives in Table 2, we can see that the sort results of x_1,x_2,x_3,x_4,x_5 will change with the increase of λ.

$Table2 :\ the\ sequence\ of\ alternatives$

	$s(x_1)$	$s(x_2)$	$s(x_3)$	$s(x_4)$	$s(x_5)$
$GHFPWA_{\frac{1}{2}}$ $x_5 \succ x_2 \succ x_1 \succ x_3 \succ x_4$	0.6286	0.6355	0.6248	0.6051	0.6461
$GHFPWA_1$ $x_5 \succ x_2 \succ x_1 \succ x_3 \succ x_4$	0.6329	0.6376	0.6306	0.6100	0.6486
$GHFPWA_2$ $x_5 \succ x_2 \succ x_1 \succ x_3 \succ x_4$	0.6420	0.6423	0.6411	0.6209	0.6536
$GHFPWA_3$ $x_5 \succ x_1 \succ x_3 \succ x_2 \succ x_4$	0.6514	0.6469	0.6497	0.6328	0.6583
$GHFPWA_5$ $x_1 \succ x_5 \succ x_3 \succ x_4 \succ x_2$	0.6695	0.6554	0.6621	0.6576	0.6669
$GHFPWA_7$ $x_1 \succ x_4 \succ x_5 \succ x_3 \succ x_2$	0.6851	0.6627	0.6703	0.6812	0.6747

Analogously, we can perform the same analysis on the QHFPWA operator, thereby, according to the different aggregated requirements under hesitant fuzzy environment, we can choose the appropriate generalized prioritized aggregation operator in solving MADM problem, which takes into account prioritization among attributes.

VI. Conclusion

In this paper, we investigate the hesitant fuzzy MADM problems in which the attributes are in different priority level. Then, based on the idea of prioritized aggregation operator [2], [5], [22] and the hesitant fuzzy prioritized aggregation operator proposed by Ref [19], we propose a different method to determine the weight vectors associated with the prioritized relationship of the aggregated arguments. In addition, we define the HFPWA operator based on the proposed method. Moreover, we develop some generalized hesitant fuzzy prioritized aggregation operator (GHFPWA,QHFPWA,HFPMWA) and investigate some of their desirable properties in detail. To reflect the priority level of the aggregated arguments, we apply these proposed generalized prioritized aggregation operators to develop a MADM method that take into account prioritization among attributes. Finally, an example is given to illustrate the effectiveness of decision-making methods, and we make further analysis on the sequence of alternative by the different generalized hesitant fuzzy prioritized aggregation operators. It is worth noting that the results of this paper can be extended to the interval hesitant fuzzy environment.

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