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AN INTUITIONISTIC FUZZY GROUP DECISION-MAKING APPROACH BASED ON ENTROPY AND SIMILARITY MEASURES

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In this paper, we study group decision-making problems based on intuitionistic preference relations. By measuring the uncertain information of intuitionistic preference relations and the average similarity degree of one individual intuitionistic preference relation to the others, we propose a new approach to assess the relative importance weights of experts. The approach takes both the objective and subjective information of experts into consideration. We then integrate the weights of experts into the individual intuitionistic preference relations and develop a relative similarity method to derive the priorities of alternatives. The comparison analysis with other methods by two numerical examples illustrates the practicality and effectiveness of the proposed methods.

Keywords: Group decision-making; intuitionistic preference relation; intuitionistic fuzzy set; entropy; similarity measure.

1. Introduction

Intuitionistic fuzzy sets (IFSs),[1] characterized by membership functions and nonmembership functions, have been applied in many fields, such as decision-making,[2–13] medical diagnosis,[14] and pattern recognition.[15,16] In group decision-making problems based on IFSs, Szimidt and Kacprzyk[3–5] investigated the extent of agreement in a group of experts (individuals) as the individual preferences are described by intuitionistic fuzzy preference relations. They gave a method to aggregate the individual intuitionistic fuzzy preference relations into a collective intuitionistic fuzzy preference relation, while the alternatives are not ranked.[6] We know that the intuitionistic fuzzy preference relations given in Refs. 3–5 consist of three types of matrices. Xu[9] combined the three types of matrices into one matrix and proposed the concept of an intuitionistic preference relation. He developed an approach to group decision-making based on intuitionistic preference relations, where an intuitionistic fuzzy arithmetic averaging operator and an intuitionistic fuzzy weighted arithmetic averaging operator are used to aggregate intuitionistic preference information.

Among most of those literatures on group decision-making problems, the importance weights of experts are usually predetermined by the experts' social status and competence recognized by their domain fields, etc., which are usually regarded as the given parameters. In this paper, such kind of given weights are referred to as the subjective weights of experts. During the decision-making process, the preferences over those alternatives, provided by the experts, also reflect their practical knowledge toward those alternatives, and are worth contributing some information toward the importance weights of the experts. Then it is quite interesting to study how to derive the weights of experts, referred to as the objective weights of experts in this paper, from their corresponding intuitionistic preference relations, which describe the experts' preference information about each pair of alternatives. An approach to assess the objective weights of experts is studied in this paper by using some information measure tools — entropy and similarity measures of IFSs.

As two important topics in the theory of fuzzy sets, both entropy and similarity measures of IFSs have been investigated widely from different points of view. Burillo and Bustince[17] introduced the notions of entropy of interval value fuzzy set (IVFS) and IFS to measure the degree of intuitionism of an IVFS or IFS. Szimidt and Kacprzyk[18] proposed a nonprobabilistic-type entropy with a geometric interpretation of IFSs. Hung and Yang[19] gave their axiom definitions of entropies of IFS and IVFS by exploiting the concept of probability. After that, many authors also proposed different entropy formulas for IFSs,[15,20] IVFSs,[21–23] interval type-2 fuzzy sets (IT2 FS),[24] and vague sets.[25–27]

The similarity measures of IFSs are used to estimate the degree of similarity between two IFSs. Li and Cheng[28] defined the notion of the degree of similarity between IFSs, and introduced several similarity measures between IFSs. Mitchell[29] made some modification to those of Li and Cheng.[28] Szmidt and Kacprzyk[7] defined a similarity measure using a distance measure, which involves both similarity and dissimilarity. Xu[30] gave a comprehensive overview of similarity measures of IFSs and defined some continuous similarity measures based on different distance measures. Li *et al.*[31] made a comparative analysis of similarity measures of IFSs. On the similarity measures for IVFSs, IT2 FSs, and vague sets, we can refer to Refs. 23 and 32–37.

On the relationship between entropy and similarity measures, it has already been proved that similarity measures for IVFSs can be transformed by entropy measures of IVFSs.[21–23] Motivated by the above-mentioned studies, in this paper, we propose a new similarity measure for IFSs, which is based on the argument about the relationship among the entropy formulas defined in Refs. 18, 20 and 26 and a transformation of entropy measures into similarity measures of IFSs. Then, by the entropy and similarity measures of intuitionistic preference relations, we propose an approach to assess the weights of experts and develop a relative similarity method to rank the alternatives.

The rest of the paper is organized as follows. Section 2 proves that three entropy formulas of IFSs given in Refs. 18, 20 and 26, respectively, are the same, and also presents an effective similarity measure for IFSs by a transformation of entropy measures into similarity measures. Section 3 introduces an approach to determine the weights of experts by their subjective and objective weights, and proposes a relative similarity method to rank the alternatives. Two examples on group decision-making problems are shown to illustrate the effectiveness and reasonability of the proposed methods by comparisons with others. Section 4 gives the conclusions.

2. Preliminaries

2.1. *Intuitionistic fuzzy entropy*

In this subsection, we review some entropy formulas and discuss their relations.

Definition 1.[1] Let X be a universe of discourse. An intuitionistic fuzzy set in X is an object having the form:

$$A = \{\langle x, u_A(x), v_A(x)\rangle \mid x \in X\}, \tag{1}$$

where the mappings

$$u_A : X \to [0,1] \quad \text{and} \quad v_A : X \to [0,1]$$

satisfy the condition

$$0 \leq u_A(x) + v_A(x) \leq 1, \quad \forall\, x \in X.$$

The numbers $u_A(x)$ and $v_A(x)$ denote the degree of membership and nonmembership of x to A, respectively.

For convenience of notations, we abbreviate "intuitionistic fuzzy set" to IFS, and denote by IFS(X) the set of all IFSs in X.

For each IFS A in X, we call $1 - u_A(x) - v_A(x)$, denoted by $\pi_A(x)$, the intuitionistic index of x in A, which denotes the hesitancy degree of x to A.[14]

Definition 2.[1] For two IFSs $A = \{\langle x, u_A(x), v_A(x)\rangle \mid x \in X\}$ and $B = \{\langle x, u_B(x), v_B(x)\rangle \mid x \in X\}$, their relations and operations are defined as follows:

(1) $A \subseteq B$ if and only if $u_A(x) \leq u_B(x)$, $v_A(x) \geq v_B(x)$, for each $x \in X$;
(2) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$;
(3) $A^C = \{\langle x, v_A(x), u_A(x)\rangle \mid x \in X\}$.

In the following, we assume that the universe X is a finite set, listed by $\{x_1, x_2, \ldots, x_n\}$.

Szmidt and Kacprzyk[18] extended the axioms for fuzzy sets proposed by De Luca and Termini[39] to introduce the concept of entropy for IFSs, and proposed a nonprobabilistic-type entropy.

Definition 3.[18] A real-valued function E: IFS$(X) \to [0,1]$ is called an entropy measure on IFS(X) if it satisfies the following axiomatic requirements:

(E1) $E(A) = 0$ if and only if A is a crisp set;
(E2) $E(A) = 1$ if and only if $u_A(x_i) = v_A(x_i)$ for each x_i in X;
(E3) $E(A) = E(A^C)$;
(E4) $E(A) \le E(B)$ if $u_A(x_i) \ge u_B(x_i)$ and $v_B(x_i) \ge v_A(x_i)$ for $u_B(x_i) \ge v_B(x_i)$, or $u_A(x_i) \le u_B(x_i)$ and $v_B(x_i) \le v_A(x_i)$ for $u_B(x_i) \le v_B(x_i)$ for any $x_i \in X$.

We call $E(A)$ the entropy of A for each A in IFS(X).

For each IFS $A = \{x_i, u_A(x_i), v_A(x_i) \,|\, x_i \in X\}$, Szmidt and Kacprzyk[18] defined the following two kinds of cardinalities of A: the least cardinality or min-sigma-count of A given by

$$\min \sum \mathrm{count}(A) = \sum_{i=1}^{n} u_A(x_i), \tag{2}$$

and the biggest cardinality or max-sigma-count of A given by

$$\max \sum \mathrm{count}(A) = \sum_{i=1}^{n} (u_A(x_i) + \pi_A(x_i)). \tag{3}$$

Using these two cardinalities, Szmidt and Kacprzyk[18] an entropy measure $E_{\mathrm{SK}}(A)$ of an IFS A as

$$E_{\mathrm{SK}}(A) = \frac{1}{n} \sum_{i=1}^{n} \frac{\max \mathrm{count}(A_i \cap A_i^C)}{\max \mathrm{count}(A_i \cup A_i^C)}, \tag{4}$$

where, for each i, A_i denotes the single-element IFS corresponding to the element x_i in X, described as $A_i = \{\langle x_i, u_A(x_i), v_A(x_i)\rangle\}$, and

$$A_i \cap A_i^C = \{\langle x_i, \min\{u_A(x_i), v_A(x_i)\}, \max\{v_A(x_i), u_A(x_i)\}\rangle\}, \tag{5}$$

$$A_i \cup A_i^C = \{\langle x_i, \max\{u_A(x_i), v_A(x_i)\}, \min\{v_A(x_i), u_A(x_i)\}\rangle\}. \tag{6}$$

Remark 1. Since both $A_i \cap A_i^C$ and $A_i \cup A_i^C$ contain only one element, the biggest cardinalities of $A_i \cap A_i^C$ and $A_i \cup A_i^C$ defined by (3) are reduced to $\max \mathrm{count}(A_i \cap A_i^C)$ and $\max \mathrm{count}(A_i \cup A_i^C)$, respectively.

After the work of Szmidt and Kacprzyk,[18] Wang and Lei[20] gave a different entropy formula by

$$E_{\mathrm{WL}}(A) = \frac{1}{n} \sum_{i=1}^{n} \frac{\min\{u_A(x_i), v_A(x_i)\} + \pi_A(x_i)}{\max\{v_A(x_i), u_A(x_i)\} + \pi_A(x_i)}. \tag{7}$$

Note that, Huang and Liu[26] introduced a fuzzy entropy for a vague set proposed by Gau and Buehrer.[39] Using the fact of the equivalence of two theories of vague sets and IFSs proved by Bustince and Burillo,[40] we can transform the fuzzy entropy formula for a vague set in Ref. 26 to an entropy formula for an IFS A by the following equation:

$$E_{\mathrm{HL}}(A) = \frac{1}{n}\sum_{i=1}^{n}\frac{1 - |u_A(x_i) - v_A(x_i)| + \pi_A(x_i)}{1 + |u_A(x_i) - v_A(x_i)| + \pi_A(x_i)}. \tag{8}$$

The entropy formulas (4), (7) and (8) are introduced from different points of view. Comparing the three formulas, we get the following theorem.

Theorem 1. *For each IFS* $A = \{\langle x, u_A(x), v_A(x)\rangle \,|\, x \in X\}$, $E_{\mathrm{SK}}(A) = E_{\mathrm{WL}}(A) = E_{\mathrm{HL}}(A)$.

Proof. Since

$$A_i \cap A_i^C = \{\langle x_i, \min\{u_A(x_i), v_A(x_i)\}, \max\{v_A(x_i), u_A(x_i)\}\rangle\}$$

and

$$A_i \cup A_i^C = \{\langle x_i, \max\{u_A(x_i), v_A(x_i)\}, \min\{v_A(x_i), u_A(x_i)\}\rangle\},$$

we can get that

$$\begin{aligned} E_{\mathrm{SK}}(A) &= \frac{1}{n}\sum_{i=1}^{n}\frac{\max\mathrm{count}(A_i \cap A_i^C)}{\max\mathrm{count}(A_i \cup A_i^C)} \\ &= \frac{1}{n}\sum_{i=1}^{n}\frac{\min\{u_A(x_i), v_A(x_i)\} + \pi_A(x_i)}{\max\{u_A(x_i), v_A(x_i)\} + \pi_A(x_i)} \\ &= E_{\mathrm{WL}}(A). \end{aligned}$$

Suppose $u_A(x_i) \geq v_A(x_i)$. Then

$$\begin{aligned} E_{\mathrm{HL}}(A) &= \frac{1}{n}\sum_{i=1}^{n}\frac{1 - |u_A(x_i) - v_A(x_i)| + \pi_A(x_i)}{1 + |u_A(x_i) - v_A(x_i)| + \pi_A(x_i)} \\ &= \frac{1}{n}\sum_{i=1}^{n}\frac{1 - u_A(x_i) + v_A(x_i) + \pi_A(x_i)}{1 + u_A(x_i) - v_A(x_i) + \pi_A(x_i)} \\ &= \frac{1}{n}\sum_{i=1}^{n}\frac{v_A(x_i) + \pi_A(x_i)}{u_A(x_i) + \pi_A(x_i)} \\ &= \frac{1}{n}\sum_{i=1}^{n}\frac{\min\{u_A(x_i), v_A(x_i)\} + \pi_A(x_i)}{\max\{v_A(x_i), u_A(x_i)\} + \pi_A(x_i)} \\ &= E_{\mathrm{WL}}(A). \end{aligned}$$

Similarly, when $u_A(x_i) \leq v_A(x_i)$, we can also obtain the same conclusion that $E_{\mathrm{WL}}(A) = E_{\mathrm{HL}}(A)$.

In the next subsection, we construct a similarity measure for IFSs by using the entropy measure E_{WL}. □

2.2. *A similarity measure for IFSs*

Definition 4.[16,30] A real-valued function $S : \mathrm{IFS}(X) \times \mathrm{IFS}(X) \to [0,1]$ is called a similarity measure on $\mathrm{IFS}(X)$, if it satisfies the following axiomatic requirements:

(S1) $0 \le S(A,B) \le 1$;
(S2) $S(A,B) = 1$ if and only if $A = B$;
(S3) $S(A,B) = S(B,A)$;
(S4) If $A \subseteq B \subseteq C$, then $S(A,C) \le S(A,B)$ and $S(A,C) \le S(B,C)$.

Zeng and Li[21] investigated the relationship between similarity measures and entropy measures of IVFSs. By the equivalence of IVFSs and IFSs,[41,42] we introduce a transforming method by which one can set up a similarity measure for IFSs based on an entropy measure.

For A and B in $\mathrm{IFS}(X)$, let

$$u_{M(A,B)}(x_i) = \frac{1 + \min\{|u_A(x_i) - u_B(x_i)|, |v_A(x_i) - v_B(x_i)|\}}{2}, \tag{9}$$

$$v_{M(A,B)}(x_i) = \frac{1 - \max\{|u_A(x_i) - u_B(x_i)|, |v_A(x_i) - v_B(x_i)|\}}{2}. \tag{10}$$

Then we define $M(A,B) = \{\langle x_i, u_{M(A,B)}(x_i), v_{M(A,B)}(x_i)\rangle | x_i \in X\}$. Obviously, $M(A,B)$ is an IFS. From Ref. 21 and the relationship of IVFs and IFSs, it is easy to get the following theorem.

Theorem 2. *Suppose that E is an entropy measure on $IFS(X)$. Then $E(M(A,B))$, for each pair of IFSs A and B, is a similarity measure on $IFS(X)$.*

Corollary 1. *Let E be the entropy measure defined by*

$$E_{\mathrm{WL}}(A) = \frac{1}{n}\sum_{i=1}^{n} \frac{\min\{u_A(x_i), v_A(x_i)\} + \pi_A(x_i)}{\max\{v_A(x_i), u_A(x_i)\} + \pi_A(x_i)} \quad \text{for } A \in IFS(X).$$

Then the function S defined by

$$S(A,B) = \frac{1}{n}\sum_{i=1}^{n} \frac{1 - \min\{|u_A(x_i) - u_B(x_i)|, |v_A(x_i) - v_B(x_i)|\}}{1 + \max\{|u_A(x_i) - u_B(x_i)|, |v_A(x_i) - v_B(x_i)|\}} \quad \text{for } A, B \in IFS(X), \tag{11}$$

is a similarity measure on $IFS(X)$.

Considering the elements in the universe may have different importance, here we define the weighted form of formula (11).

Let $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ be a weighting vector with $\omega_i \geq 0$ and $\sum_{i=1}^{n} \omega_i = 1$ of the elements x_i $(i = 1, 2, \ldots, n)$. The weighted similarity measure is defined as

$$S(A,B) = \sum_{i=1}^{n} \omega_i \frac{1 - \min\{|u_A(x_i) - u_B(x_i)|, |v_A(x_i) - v_B(x_i)|\}}{1 + \max\{|u_A(x_i) - u_B(x_i)|, |v_A(x_i) - v_B(x_i)|\}}. \tag{12}$$

Obviously, when $\omega = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$, formula (12) is reduced to formula (11).

3. Group Decision Making Based on Intuitionistic Preference Relations

In this section, the entropy and similarity measure of IFSs are applied to determine the weights of experts and rank the alternatives for group decision-making problems based on intuitionistic preference relations.

3.1. *Intuitionistic preference relations*

During the decision-making process, an expert is usually required to provide his/her preferences over the alternatives. The expert may provide his/her judgments in a certain way while sometimes he/she is not quite confident of those judgments. Thus, it is appropriate to express the expert's preference values with intuitionistic fuzzy values rather than the numerical values.[2,6,9] Szmidt and Kacprzyk[6–8] first generalized the fuzzy preference relation to the intuitionistic fuzzy preference relation consisting of three types of matrices. Later, by combining the three types of matrices into one matrix, Xu[9,10] introduced the concept of an intuitionistic preference relation.

Definition 5.[9] An intuitionistic preference relation R on the set X is represented by a matrix $R = (r_{ij})_{n\times n}$ with $r_{ij} = \langle (x_i, x_j), u(x_i, x_j), v(x_i, x_j)\rangle$ for all $i, j = 1, 2, \ldots, n$. For convenience, for all i, j, we let $r_{ij} = (u_{ij}, v_{ij})$, where r_{ij} is an intuitionistic fuzzy value consisting of the certainty degree u_{ij} to which x_i is preferred to x_j and the certainty degree v_{ij} to which x_i is nonpreferred to x_j, and u_{ij}, v_{ij} satisfy the following characteristics:

$$0 \leq u_{ij} + v_{ij} \leq 1, \quad u_{ji} = v_{ij}, \ \ v_{ji} = u_{ij}, \ \ u_{ii} = v_{ii} = 0.5 \quad \text{for all } i, j = 1, 2, \ldots, n.$$

$\pi_{ij} = 1 - u_{ij} - v_{ij}$ is interpreted as the uncertainty degree to which x_i is preferred to x_j.

Next we discuss how to acquire more information from the experts' preferences over the alternatives so as to adjust the given importance weights of experts for more reasonable decision-making.

3.2. *A method to determine the weights of experts*

The group decision-making problem considered in this paper can be described as follows: let $X = \{x_1, x_2, \ldots, x_n\}$ be the set of alternatives, $E = \{e_1, e_2, \ldots, e_m\}$

be the set of experts. The expert e_k provides his/her preference information for each pair of alternatives, and constructs an intuitionistic preference relation $R^{(k)} = (r_{ij}^{(k)})_{n\times n}$, where $r_{ij}^{(k)} = (u_{ij}^{(k)}, v_{ij}^{(k)})$, $0 \le u_{ij}^{(k)} + v_{ij}^{(k)} \le 1$, $u_{ji}^{(k)} = v_{ij}^{(k)}$, $v_{ji}^{(k)} = u_{ij}^{(k)}, u_{ii}^{(k)} = v_{ii}^{(k)} = 0.5$ for all $i, j = 1, 2, \ldots, n$.

For the group decision-making problem based on intuitionistic preference relations, the integration of the individual intuitionistic preference relations into a collective intuitionistic preference relation is expected. The relative importance weights of experts need to be incorporated into each individual intuitionistic preference relation and affect the aggregating result.

In reality, the weights of experts are related to their social positions or prestige, competence recognized for specific domains, etc. and are often predetermined for a group decision-making problem. However, the experts' judgments, such as intuitionistic preference relations generated during the problem solving process, may not always be considered even those fresh information reflects their actual knowledge on the alternatives. As the weights may play a dominant role toward the final ranking of the alternatives, then how to assign reasonable weights toward the experts during the practical decision-making process is an issue. In this paper, the predefined weights of the experts' importance are regarded as one kind of subjective weights of the experts. Compared with the predefined weights, the intuitionistic preference relations which express the experts' preference information may reflect their real understandings toward the alternatives in a more objective sense, then the weights of experts derived from their corresponding intuitionistic preference relations are referred as the objective weights of experts. How to obtain the reasonable objective weights of experts? Next, we propose an approach to assess the objective weights of experts using entropy and similarity measures of IFSs.

The entropy can measure the uncertain information of an IFS. Each intuitionistic preference relation $R^{(k)}$ $(k = 1, 2, \ldots, m)$ is actually an IFS in $X \times X$, hence we can measure its uncertain information by using the entropy measure defined by formula (7). During the decision-making process, we usually expect the uncertainty degree of the intuitionistic preference relation as small as possible for more certainty of the achieved results. Thus, the bigger the entropy of $R^{(k)}$, the smaller the weight given to the corresponding expert e_k. On the other hand, the similarity degree $S(R^{(k)}, R^{(l)})$ between any two individual intuitionistic preference relations $R^{(k)}$ and $R^{(l)}$ can be measured by formula (11). Then the average similarity degree of $R^{(k)}$ to the others can be calculated; the bigger the value, the larger the weight given to the expert e_k.

According to the above analysis, we develop the following Algorithm I to assess the objective weights of the experts.

Algorithm I

For the group decision-making problems based on intuitionistic fuzzy preference relations, we let $w^1 = (w_1^1, w_2^1, \ldots, w_m^1)$ be a subjective weighting vector of experts, where $w_k^1 > 0, k = 1, 2, \ldots, m, \sum_{i=1}^m w_i^1 = 1$.

Step 1. Calculate the entropy $E_{\mathrm{WL}}(R^{(k)})$ of $R^{(k)}$:

$$E_{\mathrm{WL}}(R^{(k)}) = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\min\{u_{ij}^{(k)}, v_{ij}^{(k)}\} + \pi_{ij}^{(k)}}{\max\{u_{ij}^{(k)}, v_{ij}^{(k)}\} + \pi_{ij}^{(k)}}, \quad k = 1, 2, \ldots, m. \tag{13}$$

Since $u_{ji}^{(k)} = v_{ij}^{(k)}$, $v_{ji}^{(k)} = u_{ij}^{(k)}$, $u_{ii}^{(k)} = v_{ii}^{(k)} = 0.5$ for all $i, j = 1, 2, \ldots, n$, we have

$$E_{\mathrm{WL}}(R^{(k)}) = \frac{1}{n} + \frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\frac{\min\{u_{ij}^{(k)}, v_{ij}^{(k)}\} + \pi_{ij}^{(k)}}{\max\{u_{ij}^{(k)}, v_{ij}^{(k)}\} + \pi_{ij}^{(k)}}, \quad k = 1, 2, \ldots, m. \tag{14}$$

Step 2. Calculate the weight w_k^a, determined by $E_{\mathrm{WL}}(R^{(k)})$, of the expert e_k:

$$w_k^a = \frac{1 - E_{\mathrm{WL}}(R^{(k)})}{\sum_{i=1}^{m}(1 - E_{\mathrm{WL}}(R^{(i)}))}, \quad k = 1, 2, \ldots, m. \tag{15}$$

Step 3. Calculate the similarity measure $S(R^{(k)}, R^{(l)})$ between $R^{(k)}$ and $R^{(l)}$ for each $k \neq l$:

$$S(R^{(k)}, R^{(l)}) = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1 - \min\{|u_{ij}^{(k)} - u_{ij}^{(l)}|, |v_{ij}^{(k)} - v_{ij}^{(l)}|\}}{1 + \max\{|u_{ij}^{(k)} - u_{ij}^{(l)}|, |v_{ij}^{(k)} - v_{ij}^{(l)}|\}}. \tag{16}$$

Obviously,

$$S(R^{(k)}, R^{(l)}) = \frac{1}{n} + \frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\frac{1 - \min\{|u_{ij}^{(k)} - u_{ij}^{(l)}|, |v_{ij}^{(k)} - v_{ij}^{(l)}|\}}{1 + \max\{|u_{ij}^{(k)} - u_{ij}^{(l)}|, |v_{ij}^{(k)} - v_{ij}^{(l)}|\}}. \tag{17}$$

Then the average similarity degree $S(R^{(k)})$ of $R^{(k)}$ to the others is calculated by

$$S(R^{(k)}) = \frac{1}{m-1}\sum_{l=1, l\neq k}^{m} S(R^{(k)}, R^{(l)}), \quad k = 1, 2, \ldots, m. \tag{18}$$

Step 4. Calculate the weight w_k^b determined by $S(R^{(k)})$ of the expert e_k:

$$w_k^b = \frac{S(R^{(k)})}{\sum_{i=1}^{m} S(R^{(i)})}, \quad k = 1, 2, \ldots, m. \tag{19}$$

Step 5. Calculate the objective weight w_k^2 of the expert e_k:

$$w_k^2 = \eta w_k^a + (1 - \eta) w_k^b, \quad \eta \in [0, 1], \quad k = 1, 2, \ldots, m. \tag{20}$$

Step 6. Integrate the subjective weight w_k^1 and the objective weight w_k^2 into the weight w_k of the expert e_k:

$$w_k = \gamma w_k^1 + (1 - \gamma) w_k^2, \quad \gamma \in [0, 1], \quad k = 1, 2, \ldots, m. \tag{21}$$

From Algorithm I, the weights of experts consider both the subjective and objective information. The decision-maker determines the value of γ according to his/her preferences to the objective and subjective weight information. We then integrate the individual intuitionistic preference relations into a collective intuitionistic preference relation by using the following theorem given by Ref. 10.

Theorem 3.[10] *Let $R^{(k)} = (r_{ij}^{(k)})_{n\times n}$ $(k = 1, 2, \ldots, m)$ be m intuitionistic fuzzy preference relations given by the experts $e_k, k = 1, 2, \ldots, m$, and $w = (w_1, w_2, \ldots, w_n)$ be a weighting vector of experts, where $r_{ij}^{(k)} = (u_{ij}^{(k)}, v_{ij}^{(k)})$, $w_k > 0, k = 1, 2, \ldots, m, \sum_{i=1}^{m} w_i = 1$. Then the aggregation $R = (r_{ij})_{n\times n}$ of $R^{(k)} = (r_{ij}^{(k)})_{n\times n}$ $(k = 1, 2, \ldots, m)$ is also an intuitionistic preference relation, where*

$$r_{ij} = (u_{ij}, v_{ij}), \quad u_{ij} = \sum_{k=1}^{m} w_k u_{ij}^{(k)}, \quad v_{ij} = \sum_{k=1}^{m} w_k v_{ij}^{(k)}, \quad \textit{for all } i, j = 1, 2, \ldots, n.$$

3.3. *A relative similarity method to rank the alternatives*

The ith row vector $\{(u_{ij}, v_{ij})|j = 1, 2, \ldots, n\}$ $(i = 1, 2, \ldots, n)$ of a collective intuitionistic preference relation R, denoted by R^i, describes the pairwise comparison preference of the ith alternative x_i over all the alternatives in X, and can be regarded as an IFS in $\{x_i\} \times X$. Let x^+ and x^- be the positive ideal alternative and negative ideal alternative, respectively. Suppose the intuitionistic fuzzy sets $R^+ = \{(1, 0), (1, 0), \ldots, (1, 0)\}$ and $R^- = \{(0, 1), (0, 1), \ldots, (0, 1)\}$ describe the pairwise comparison preference of x^+ and x^- over all the alternatives in X, respectively. Then the best alternative is acquired to have the degree of similarity to x^+ as big as possible and have the degree of similarity to x^- as small as possible. Thus we can rank the alternatives from the collective preference relation by using the following relative similarity method: Algorithm II. Assume that $R^{(k)}$ $(k = 1, 2, \ldots, m)$ and w are defined as before.

Algorithm II

Step 1. Calculate the collective intuitionistic preference relation $R = (r_{ij})_{n\times n}$ by

$$r_{ij} = (u_{ij}, v_{ij}) = \left(\sum_{k=1}^{m} w_k u_{ij}^{(k)}, \sum_{k=1}^{m} w_k v_{ij}^{(k)}\right), \quad i, j = 1, 2, \ldots, n. \tag{22}$$

Step 2. For each alternative x_i, calculate the similarity measure $S(R^i, R^+)$ between R^i and R^+ and the similarity measure $S(R^i, R^-)$ between R^i and R^- by formula (11). Then

$$S(R^i, R^+) = \frac{1}{n}\sum_{j=1}^{n} \frac{1 - \min\{|u_{ij} - 1|, |v_{ij} - 0|\}}{1 + \max\{|u_{ij} - 1|, |v_{ij} - 0|\}} = \frac{1}{n}\sum_{j=1}^{n} \frac{1 - \min\{1 - u_{ij}, v_{ij}\}}{1 + \max\{1 - u_{ij}, v_{ij}\}} \tag{23}$$

and

$$S(R^i, R^-) = \frac{1}{n}\sum_{j=1}^{n}\frac{1-\min\{|u_{ij}-0|, |v_{ij}-1|\}}{1+\max\{|u_{ij}-0|, |v_{ij}-1|\}} = \frac{1}{n}\sum_{j=1}^{n}\frac{1-\min\{u_{ij}, 1-v_{ij}\}}{1+\max\{u_{ij}, 1-v_{ij}\}}. \tag{24}$$

Step 3. For each alternative x_i, calculate its evaluation value

$$f(x_i) = \frac{S(R^i, R^+)}{S(R^i, R^+) + S(R^i, R^-)}. \tag{25}$$

The larger the value of $f(x_i)$, the better the alternative x_i. Then the rank of the alternatives is acquired. The following two examples are given to show how to achieve the integrated weights by Algorithm I and how to rank the alternatives by Algorithm II.

3.4. *Examples*

Here we show two examples by adopting one available example used by Xu and Yager[10] and another one by Gong *et al.*[43] Through comparison with the methods in Refs. 10 and 43, we try to show our methods expose more information which is not shown before.

Example 1. Assume that we have four alternatives x_i $(i = 1, 2, 3, 4)$, and three experts e_k $(k = 1, 2, 3)$ in a group decision-making problem. Suppose the weights for each expert are 0.5, 0.3, and 0.2, respectively. Each expert e_k $(k = 1, 2, 3)$ compares the four alternatives and constructs the intuitionistic preference relations $R^{(k)} = (r_{ij}^{(k)})_{4\times 4}$ $(k = 1, 2, 3)$, respectively, shown as follows:

$$R^{(1)} = \begin{pmatrix} (0.5, 0.5) & (0.2, 0.4) & (0.5, 0.4) & (0.7, 0.1) \\ (0.4, 0.2) & (0.5, 0.5) & (0.3, 0.5) & (0.4, 0.5) \\ (0.4, 0.5) & (0.5, 0.3) & (0.5, 0.5) & (0.8, 0.2) \\ (0.1, 0.7) & (0.5, 0.4) & (0.2, 0.8) & (0.5, 0.5) \end{pmatrix},$$

$$R^{(2)} = \begin{pmatrix} (0.5, 0.5) & (0.3, 0.4) & (0.4, 0.5) & (0.6, 0.3) \\ (0.4, 0.3) & (0.5, 0.5) & (0.4, 0.4) & (0.5, 0.3) \\ (0.5, 0.4) & (0.4, 0.4) & (0.5, 0.5) & (0.7, 0.2) \\ (0.3, 0.6) & (0.3, 0.5) & (0.2, 0.7) & (0.5, 0.5) \end{pmatrix},$$

$$R^{(3)} = \begin{pmatrix} (0.5, 0.5) & (0.8, 0.1) & (0.3, 0.4) & (0.6, 0.4) \\ (0.1, 0.8) & (0.5, 0.5) & (0.5, 0.3) & (0.4, 0.5) \\ (0.4, 0.3) & (0.3, 0.5) & (0.5, 0.5) & (0.3, 0.7) \\ (0.4, 0.6) & (0.5, 0.4) & (0.7, 0.3) & (0.5, 0.5) \end{pmatrix}.$$

Example 1 was adopted by Xu and Yager[10] for consensus analysis in group decision-making based on intuitionistic preference relations. Here we use the data to derive the ranking order of the alternatives.

The original weights $0.5, 0.3, 0.2$ of the three experts are regarded as the subjective weights, i.e. we suppose the subjective weighting vector w^1 is $(0.5, 0.3, 0.2)$. Next we derive the objective weighting vector of the experts and aggregate the subjective and objective weighting vectors into the integrated weighting vector of the experts by Algorithm I.

By formula (14), we get the entropies of $R^{(i)}$ $(i = 1, 2, 3)$:

$$E_{\mathrm{WL}}(R^{(1)}) = 0.7143, \quad E_{\mathrm{WL}}(R^{(2)}) = 0.7939, \quad E_{\mathrm{WL}}(R^{(3)}) = 0.7153.$$

Then, by formula (15), we get the weighting vector $w^a = (0.3679, 0.2654, 0.3666)$ of the experts determined by the entropies.

Using formula (17), we have

$$S(R^{(1)}, R^{(2)}) = 0.8693, \quad S(R^{(1)}, R^{(3)}) = 0.7454, \quad S(R^{(2)}, R^{(3)}) = 0.7703.$$

Then, by formulas (18) and (19), we get the averaged similarity degrees $S(R^{(i)})$ of $R^{(i)}$ $(i = 1, 2, 3)$ and the weighting vector w^b of the experts determined by average similarity degrees, respectively:

$$S(R^{(1)}) = 0.8074, \quad S(R^{(2)}) = 0.8198, \quad S(R^{(3)}) = 0.7578,$$
$$w^b = (0.3386, 0.3438, 0.3178).$$

Let $\eta = 0.5$, which means either the weight determined by the entropy or the weight by the similarity measure contributes half to the objective weight. By formula (20), we get the objective weighting vector

$$w^2 = (0.3533, 0.3046, 0.3422).$$

We can integrate the subjective weighting vector w^1 and the objective weighting vector w^2 into the integrated weighting vector w by formula (21): $w = \gamma w^1 + (1 - \gamma) w^2, \gamma \in [0, 1]$, where γ is determined by the decision-makers according to their preferences to the objective and subjective weight information. Here at first we suppose $\gamma = 0.5$ in formula (21) and obtain the integrated weighting vector w:

$$w = (0.4266, 0.3023, 0.2711).$$

Till now, we have obtained the integrated weights of experts for the practical decision-making problem. Next, we consider to integrate the weighting vector $w = (0.4266, 0.3023, 0.2711)$ into the intuitionistic preference relations of experts and derive the ranking of alternatives.

First, we describe Xu's approach[9] to derive the decision result, which involves the following steps:

Step 1. Use the formula: $r_i^{(k)} = \frac{1}{n}\sum_{j=1}^{n} r_{ij}^{(k)}, i = 1, 2, \ldots, n$, to get the averaged intuitionistic fuzzy value $r_i^{(k)}$ of the alternative x_i over all the other alternatives:

$$r_1^{(1)} = (0.475, 0.35), \quad r_2^{(1)} = (0.4, 0.425),$$
$$r_3^{(1)} = (0.55, 0.375), \quad r_4^{(1)} = (0.325, 0.6),$$
$$r_1^{(2)} = (0.45, 0.425), \quad r_2^{(2)} = (0.45, 0.375),$$
$$r_3^{(2)} = (0.525, 0.375), \quad r_4^{(2)} = (0.325, 0.575),$$
$$r_1^{(3)} = (0.55, 0.35), \quad r_2^{(3)} = (0.375, 0.525),$$
$$r_3^{(3)} = (0.375, 0.5), \quad r_4^{(3)} = (0.525, 0.45).$$

Step 2. Use the formula: $r_i = \sum_{k=1}^{m} w_k r_i^{(k)}, i = 1, 2, \ldots, n$, to aggregate all $r_i^{(k)}$, $(k = 1, 2, \ldots, m)$, corresponding to m experts, into a collective intuitionistic fuzzy value $r_i = (u_i, v_i)$ of the alternative x_i over all the other alternatives:

$$r_1 = (0.4878, 0.34727), \quad r_2 = (0.4083, 0.4370), \quad r_3 = (0.4950, 0.4089),$$
$$r_4 = (0.3792, 0.5518).$$

Step 3. Calculate the score function $S(r_i) = u_i - v_i$ of r_i and get

$$S(r_1) = 0.1151, \quad S(r_2) = -0.0287, \quad S(r_3) = 0.0861, \quad S(r_4) = -0.1726.$$

Then

$$S(r_1) > S(r_3) > S(r_2) > S(r_4),$$

and hence

$$x_1 \succ x_3 \succ x_2 \succ x_4,$$

where the notation $\succ$ indicates that one alternative is preferred to another.

Now we give the ranking result derived by our relative similarity method. By (22) in Algorithm II, we get the collective intuitionistic preference relation

$$R = \begin{pmatrix} (0.5000, 0.5000) & (0.3929, 0.3187) & (0.4156, 0.4303) & (0.6427, 0.2418) \\ (0.3187, 0.3929) & (0.5000, 0.5000) & (0.3845, 0.4156) & (0.4303, 0.4396) \\ (0.4303, 0.4156) & (0.4156, 0.3845) & (0.5000, 0.5000) & (0.6342, 0.3356) \\ (0.2418, 0.6427) & (0.4396, 0.4303) & (0.3356, 0.6342) & (0.5000, 0.5000) \end{pmatrix}.$$

By formulas (23) and (24), we obtain

$$S(R^1, R^+) = 0.4189, \quad S(R^2, R^+) = 0.3533,$$
$$S(R^3, R^+) = 0.3952, \quad S(R^4, R^+) = 0.2804,$$

$$S(R^1, R^-) = 0.3010, \quad S(R^2, R^-) = 0.2761,$$
$$S(R^3, R^-) = 0.3108, \quad S(R^4, R^-) = 0.3400.$$

Then formula (25) gives the evaluation values of alternatives x_i $(i = 1, 2, 3, 4)$:

$$f(x_1) = 0.5818, \quad f(x_2) = 0.5614, \quad f(x_3) = 0.5597, \quad f(x_4) = 0.4519.$$

Since

$$f(x_1) > f(x_2) > f(x_3) > f(x_4),$$

we have

$$x_1 \succ x_2 \succ x_3 \succ x_4.$$

By Algorithm II and Xu's approach,[9] we have that x_1 ranks the top, x_4 ranks the last, however, x_2 and x_3 have different ranking order. The ranking orders of alternatives by both methods are a little different, but the advantage that x_2 over x_3, or vice versa, is not so bigger than that of the other pairs of the alternatives, since the deviation of $f(x_2)$ and $f(x_3)$ in Algorithm II is 0.0017. Next, we go further to compare the ranking results obtained by the two methods by assigning γ with different values which show the different portions of the subjective weight and the objective weight in the total weight of the expert.

Table 1 presents the computation results by Xu's approach[9] for different weighting vectors of experts corresponding to different values of γ, and Table 2 lists the results for Algorithm II.

By comparing the ranking results as listed in both Tables 1 and 2, we find that Xu'approach[9] and Algorithm II give the same ranking orders for $\gamma = 0.2$ and $\gamma = 0$. However, for $\gamma = 1, \gamma = 0.8$, and $\gamma = 0.5$, the ranking orders by both methods are different only for the alternatives x_2 and x_3. The results given by Algorithm II are changed along with the different portions of objective and subjective parts in the

Table 1. Ranking order of the 4 alternatives under different values of γ by Xu's approach.

γ	w	r_1	r_2	r_3
1	(0.5, 0.3, 0.2)	(0.4825, 0.3725)	(0.4100, 0.4300)	(0.5075, 0.4000)
0.8	(0.4707, 0.3009, 0.2284)	(0.4846, 0.3726)	(0.4093, 0.4328)	(0.5025, 0.4036)
0.5	(0.4266, 0.3023, 0.2711)	(0.4878, 0.3727)	(0.4084, 0.4370)	(0.4950, 0.4089)
0.2	(0.3826, 0.3037, 0.3137)	(0.4910, 0.3728)	(0.4074, 0.4412)	(0.4875, 0.4143)
0	(0.3533, 0.3046, 0.3422)	(0.4931, 0.3729)	(0.4067, 0.4440)	(0.4826, 0.4178)

r_4	$S(r_1)$	$S(r_2)$	$S(r_3)$	$S(r_4)$	Raking Order
(0.3650, 0.5625)	0.1100	−0.0200	0.1075	−0.1975	$x_1 \succ x_3 \succ x_2 \succ x_4$
(0.3707, 0.5582)	0.1120	−0.0235	0.0989	−0.1875	$x_1 \succ x_3 \succ x_2 \succ x_4$
(0.3792, 0.5518)	0.1151	−0.0287	0.0861	−0.1726	$x_1 \succ x_3 \succ x_2 \succ x_4$
(0.3878, 0.5454)	0.1182	−0.0339	0.0733	−0.1576	$x_1 \succ x_3 \succ x_2 \succ x_4$
(0.3935, 0.5411)	0.1202	−0.0373	0.0647	−0.1476	$x_1 \succ x_3 \succ x_2 \succ x_4$

Table 2. Ranking order under different values of γ by Algorithm II.

γ	w	$S(R^i, R^+)$				$S(R^i, R^-)$				$f(x_1)$	$f(x_2)$	$f(x_3)$	$f(x_4)$	Ranking Order
1	(0.5, 0.3, 0.2)	0.4185	0.3578	0.4060	0.2724	0.2958	0.2721	0.3163	0.3432	0.5859	0.5681	0.5621	0.4425	$x_1 \succ x_2 \succ x_3 \succ x_4$
0.8	(0.4707, 0.3009, 0.2284)	0.4186	0.3560	0.4016	0.2755	0.2979	0.2737	0.3141	0.3419	0.5842	0.5653	0.5611	0.4462	$x_1 \succ x_2 \succ x_3 \succ x_4$
0.5	(0.4266, 0.3023, 0.2711)	0.4189	0.3533	0.3952	0.2804	0.3010	0.2761	0.3108	0.3400	0.5818	0.5614	0.5597	0.4519	$x_1 \succ x_2 \succ x_3 \succ x_4$
0.2	(0.3826, 0.3037, 0.3137)	0.4193	0.3507	0.3889	0.2853	0.3041	0.2784	0.3074	0.3380	0.5796	0.5575	0.5585	0.4577	$x_1 \succ x_3 \succ x_2 \succ x_4$
0	(0.3533, 0.3046, 0.3422)	0.4196	0.3490	0.3849	0.2887	0.3061	0.2799	0.3052	0.3366	0.5782	0.5549	0.5578	0.4616	$x_1 \succ x_3 \succ x_2 \succ x_4$

total weight of an expert. We may understand that the ranking results vary with different weights among the experts. The results of Algorithm II reflect the impacts of the changes of the weights.

Next, we show another example to compare Algorithm II with other methods.

Example 2. We consider an example that a decision-maker (potential buyer) invites three experts to help him buy a house. Suppose that the weights for each expert are 0.3, 0.4, 0.3, respectively. There are three alternatives (houses) $X = \{x_1, x_2, x_3\}$ to be chosen. The intuitionistic preference relations R_i presented by the ith expert, $i = 1, 2, 3$, are as follows:

$$R^{(1)} = \begin{pmatrix} (0.5, 0.5) & (0.1, 0.6) & (0.6, 0.3) \\ (0.6, 0.1) & (0.5, 0.5) & (0.8, 0.2) \\ (0.3, 0.6) & (0.2, 0.8) & (0.5, 0.5) \end{pmatrix},$$

$$R^{(2)} = \begin{pmatrix} (0.5, 0.5) & (0.3, 0.7) & (0.7, 0.2) \\ (0.7, 0.3) & (0.5, 0.5) & (0.6, 0.2) \\ (0.2, 0.7) & (0.2, 0.6) & (0.5, 0.5) \end{pmatrix},$$

$$R^{(3)} = \begin{pmatrix} (0.5, 0.5) & (0.3, 0.6) & (0.9, 0.1) \\ (0.6, 0.3) & (0.5, 0.5) & (0.7, 0.2) \\ (0.1, 0.9) & (0.2, 0.7) & (0.5, 0.5) \end{pmatrix}.$$

Example 2 was adopted by Gong *et al.*[43] to illustrate their goal programming approach to obtain the priority vectors from the intuitionistic preference relations. By their method, the intuitionistic fuzzy values of alternatives x_1, x_2, and x_3 are $(0.2599, 0.6519), (0.4946, 0.3632)$, and $(0.1033, 0.8426)$, respectively. The corresponding score function values are $-0.3920, 0.1314$, and -0.7393, respectively. Thus, the ranking order of the alternatives is $x_2 \succ x_1 \succ x_3$.

Next, we derive the decision results by Xu's approach[9] and Algorithm II for the given weighting vector $w = (0.3, 0.4, 0.3)$.

We first calculate the decision result by using Xu's approach.[9] By the formula $r_i^{(k)} = \frac{1}{n}\sum_{j=1}^{n} r_{ij}^{(k)}$, we get the averaged intuitionistic fuzzy values $r_i^{(k)}$ of the alternatives x_i over all the other alternatives:

$$r_1^{(1)} = (0.4, 0.4667), \quad r_2^{(1)} = (0.6333, 0.2667), \quad r_3^{(1)} = (0.3333, 0.6333),$$
$$r_1^{(2)} = (0.5, 0.4667), \quad r_2^{(2)} = (0.6, 0.3333), \quad r_3^{(2)} = (0.3, 0.6),$$
$$r_1^{(3)} = (0.5667, 0.4), \quad r_2^{(3)} = (0.6, 0.3333), \quad r_3^{(3)} = (0.2667, 0.7).$$

Using the formula $r_i = \sum_{k=1}^{m} w_k r_i^{(k)}$, we obtain the collective intuitionistic fuzzy values $r_i = (u_i, v_i)$ of the alternatives x_i over all the other alternatives:

$$r_1 = (0.4900, 0.4467), \quad r_2 = (0.6100, 0.3133), \quad r_3 = (0.3000, 0.6400).$$

Then we get the score functions

$$S(r_1) = 0.0433, \quad S(r_2) = 0.2967, \quad S(r_3) = -0.3400.$$

Since $S(r_2) > S(r_1) > S(r_3)$, we have $x_2 \succ x_1 \succ x_3$.

Next, we calculate the result by Algorithm II. By formula (22) in Algorithm II, we get the collective intuitionistic preference relation

$$R = \begin{pmatrix} (0.50, 0.50) & (0.24, 0.64) & (0.73, 0.20) \\ (0.64, 0.24) & (0.50, 0.50) & (0.69, 0.20) \\ (0.20, 0.73) & (0.20, 0.69) & (0.50, 0.50) \end{pmatrix}.$$

By (23), (24), and (25), we get the evaluation values $f(x_i)$ of alternatives x_i $(i = 1, 2, 3)$:

$$f(x_1) = \frac{0.3892}{0.3892 + 0.3474} = 0.5284, \quad f(x_2) = \frac{0.4979}{0.4979 + 0.2367} = 0.6778,$$
$$f(x_3) = \frac{0.2185}{0.2185 + 0.5246} = 0.2940.$$

Then $f(x_2) > f(x_1) > f(x_3)$, and so, $x_2 \succ x_1 \succ x_3$. Hence Algorithm II, Xu's approach,[9] and Gong's approach[43] give the same ranking results in Example 2.

As the theories of IFSs and IVFSs are equivalent,[41,42] we can apply the methods of interval fuzzy preference relation to rank the alternatives by some transformations. Here for example, we transform the collective intuitionistic preference relation R to an interval fuzzy preference relation R':

$$R' = \begin{pmatrix} [0.50, 0.50] & [0.24, 0.36] & [0.73, 0.80] \\ [0.64, 0.76] & [0.50, 0.50] & [0.69, 0.80] \\ [0.20, 0.27] & [0.20, 0.31] & [0.50, 0.50] \end{pmatrix}.$$

For the interval fuzzy preference relations, Xu and Chen[44] presented some linear programming models to derive the interval priority weights of alternatives based on the additive transitivity or the multiplicative transitivity. Now we give the decision results for the interval fuzzy preference relation R' by Xu and Chen's methods.[44]

Using the models (M-3), (M-4), and (M-5) based on the additive transitivity in Ref. 45, we get the interval priority weights ω_i of alternatives x_i $(i = 1, 2, 3)$:

$$\omega_1 = [0.3639, 0.3645], \quad \omega_2 = [0.6268, 0.6275], \quad \omega_3 = [0.0083, 0.0009].$$

Then we can construct the possibility degree matrix P by the method in Ref. 44: $P = \begin{pmatrix} 0.5 & 0 & 1 \\ 1 & 0.5 & 1 \\ 0 & 0 & 0.5 \end{pmatrix}$. Summing all entries in each row of P, we have $p_1 = 1.5$, $p_2 = 2.5$, $p_3 = 0.5$. Thus $\omega_2 \overset{100\%}{>} \omega_1 \overset{100\%}{>} \omega_3$, which indicates that $x_2 \succ x_1 \succ x_3$.

Using the models (M-11), (M-12), and (M-13) based on the multiplicative transitivity in Ref. 44, we get the interval priority weights ω_i of alternatives x_i $(i = 1, 2, 3)$:

$$\omega_1 = [0.3103, 0.3103], \quad \omega_2 = [0.5517, 0.5517], \quad \omega_3 = [0.1379, 0.1379].$$

So we have $x_2 \succ x_1 \succ x_3$. Hence the ranking results derived by Xu and Chen's methods[44] are the same as that by Algorithm II.

Above two examples illustrate that Algorithm II is a reasonable method to rank alternatives in practice. Moreover, the evaluation values given by formula (25) in Algorithm II may also serve as some kind of weights of alternatives for further use, while the others (Xu's approach,[9] Gong's approach,[43] and Xu and Chen's methods[44]) do not provide such kind of information, since the evaluation values of the alternatives acquired by those three methods are either intuitionistic fuzzy values or interval values instead of crisp values. On the other hand, we may use Algorithm I to obtain the objective weights of experts according to the intuitionistic preference relations of the experts. A decision-maker can choose proper values of the parameter γ according to his/her preference to the subjective or objective weight information of experts under practical circumstances. Thus, by integrating Algorithm I and Algorithm II, we can deal with group decision-making problems based on the intuitionistic preference relations more flexibly and effectively.

4. Conclusion

Recently, many similarity measures and entropy formulas have been applied to the group decision-making problems based on intuitionistic fuzzy information. In this paper, we apply the measures on entropy and similarity to measure the uncertain information of intuitionistic preference relations and the average similarity degree of one individual intuitionistic preference relation to the others, respectively. We develop a method to assess the importance weights of experts by taking into account both the subjective and objective weights of the experts. The subjective part of the weights denotes to the traditional weights usually predetermined according to the experts' social or academic fame or administrative positions in reality. We extract information from the experts' practical judgments (individual intuitionistic preference relations) toward the alternatives, and transform it into the objective weights of the experts by Algorithm I. We also aggregate the individual intuitionistic preference relations into a collective intuitionistic preference relation by using an intuitionistic fuzzy weighted arithmetic averaging operator, and propose a relative similarity method to derive the priorities of alternatives from the collective intuitionistic preference relation by Algorithm II.

Two examples are exhibited to compare our methods with some others to show the feasibility of our methods. And the evaluation value of the alternatives derived from Algorithm II may also be used as a kind of weights of the alternatives for further use.

Some future work may involve the study on the other similarity measures and entropy formulas for IFSs, as well as their comparative analysis and applications in multiple-criteria fuzzy group decision-making problems.

Even we go forward to consider more fresh information (the experts' judgments toward the alternatives) to adjust the weights of the experts, we find that such kind of work may still not be perfect to avoid the misuse of the weights of experts during the practical decision-making. Even adjusting the parameters, such as γ, cannot overcome the disadvantages of the methods only based on entropy and similarity measures, especially if the evidences of decision-making only come from the experts' judgments with no consideration to the evaluations of the performances of the experts. That is also a big challenge to decision-making.

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