Operators and Comparisons of Hesitant Fuzzy Linguistic Term Sets

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***Abstract*—The theory of hesitant fuzzy linguistic term sets (HFLTSs) is very useful in objectively dealing with situations in which people are hesitant in providing linguistic assessments. The purpose of this paper is to develop comparison methods and study the aggregation theory for HFLTSs. We first define operations on HFLTSs and give possibility degree formulas for comparing HFLTSs. We then define two aggregation operators for HFLTSs: a hesitant fuzzy LWA operator and a hesitant fuzzy LOWA operator. In actual application, we use these operators and the comparison methods to deal with multicriteria decision-making problems with different situations in which importance weights of criteria or experts are known or unknown.**

***Index Terms*—Hesitant fuzzy linguistic term sets (HFLTSs), multicriteria decision making (MCDM), possibility degree formula.**

I. Introduction

MANY criteria in multicriteria decision making (MCDM) are qualitative in nature. Therefore, it is more suitable to evaluate them in linguistic forms. For example, when evaluating the safety or comfort of a car, experts prefer to use fuzzy linguistic expressions such as "excellent," "good," or "poor." The fuzzy linguistic approach is a tool which has been used for modeling qualitative information in a problem [39]. Up to now, there have been many linguistic models which aim to extend and improve the fuzzy linguistic approach in information modeling and computing processes. Among them, the semantic model [1], [6], the symbolic model [7], [10], [30], and the linguistic two-tuple model [11], [12] are three classical linguistic computational models, which have been successfully applied to many areas, such as decision making [2], [13]–[15], [19], [25], [31], [38], information retrieval [3], [16], [17], supply chain management [4], [5], safety and cost analysis [18], and health care system [29].

For MCDM problems with linguistic information, a key point is how to aggregate linguistic satisfactions of an alternative under individual criteria for obtaining its overall evaluation value. Therefore, many operators have been introdued to aggregate linguistic information. Among these operators, the linguistic ordered weighted averaging (LOWA) operator, defined by Herrera *et al.* [10], was based on the OWA operator in [36] and the convex combination of linguistic terms in [7]. In [35], Yager used a linguistic-weighted median (LWM) operator to aggregate linguistic arguments and their numerical weights. In [9], Herrera and Herrera-Viedma defined a linguistic weighted averaging (LWA) operator to aggregate linguistic arguments and their linguistic weights. In order to combine the advantages of the LOWA and the LWA operators, Torra [24] defined a linguistic-weighted OWA (LWOWA) operator. For the theory of aggregation operators, see the comprehensive paper [32].

The aforementioned aggregation operators are used to aggregate single linguistic terms in a linguistic term set. However, when an expert is hesitant and thinking of several terms at the same time to assess an indicator, alternative, variable, etc., it is not easy for him/her to provide a single term as an expression of his/her knowledge. In order to model this situation, Rodríguez *et al.* [20] used Torra's idea in defining hesitant fuzzy sets [22], [23] to introduce the concept of hesitant fuzzy linguistic term sets (HFLTSs). Then, the problem of how to effectively aggregate linguistic information modeled by HFLTSs, arises and needs to be addressed. Rodríguez *et al.* [20] defined min_upper and max_lower operators to carry out the aggregation for HFLTSs. However, both operators cannot deal with the situation where the importance weights of criteria or experts are to be considered.

As to the comparisons of HFLTSs, Rodríguez *et al.* [20] gave a method for ranking HFLTSs. We note that Rodríguez's comparison method is conducted by interval values constructed by the indexes of the HFLTSs' envelopes. However, the comparison results that have been derived by this method may not accord with common sense, because it seems to be unreasonable to say one HFLTS is absolutely superior to another if these two HFLTSs have some common elements. For example, let $S = \{s_0\colon \text{ nothing}, s_1\colon \text{ very low}, s_2\colon \text{ low}, s_3\colon \text{ medium}, s_4\colon \text{ high}, s_5\colon$ very high, $s_6\colon$ perfect$\}$ be a linguistic term set. Suppose that the assessments of two cars A and B under criterion "comfort" are represented by HFLTSs $H_S^1 = \{s_3, s_4, s_5\}$ and $H_S^2 = \{s_2, s_3\}$ on S, respectively. Then, s_3 is a possible lingustic term for assessments of the two cars; therefore, car A is not absolutely better than car B under criterion "comfort." However, the method in [20] shows that H_S^1 is absolutely superior to H_S^2, which means car A is absolutely better than car B under criterion "comfort." Since HFLTSs have finite linguistic terms, the comparison methods for numerical intervals could not be directly used to compare HFLTSs.

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Our interest here is in developing new suitable comparison methods for HFLTSs, and studying the aggregation theory to deal with wider information that involves the weights of HFLTS arguments. In this paper, we use the probability theory to construct possibility degree formulas for comparing HFLTSs. Our comparison methods overcome the shortcoming explicit in the use of the comparison method in [20]. On the aggregation of HFLTS information, we introduce an HLWA operator and an HLOWA operator by defining a combination operation of HFLTSs. The HLWA operator can be used to aggregate HFLTS arguments and their numerical weights, while the HLOWA operator can aggregate HFLTS arguments and the weights associated with the arguments' ordered positions. These weights can be obtained according to the aggregation requirements of a decision maker for these arguments. Using these operators and the comparisons for HFLTSs, we introduce some decision-making methods to deal with MCDM problems. The methods can be applied to different situations, where importance weights of criteria or experts are known or unknown.

This paper is organized as follows. Section II briefly reviews some preliminary concepts that will be used in our study. Section III defines three basic operations on HFLTSs and discusses their properties. In Section IV, two possibility degree formulas are defined for ranking HFLTSs. Section V develops some aggregation operators and introduces some MCDM methods that are based on the operators and the possibility degree method. Examples are also shown to illustrate the effectiveness and reasonability of the proposed methods. In Section VI conclusions are given. The Appendix of this paper presents a possibility degree method for ranking n HFLTSs.

II. Preliminaries

In this section, we review the notations and some basic operations of HFLTSs.

We consider a finite and totally ordered linguistic term set $S = \{s_0, s_1, \ldots, s_g\}$ with odd cardinality and the midterm representing an assessment of "approximately 0.5," and with the rest of the terms being placed symmetrically around it as in [1], [6], [7], [13], and [38]. We also assume that the limit of cardinality is 11 or at most 13 [1], [13], [38]. For example, a set S of seven terms could be given as follows: $S = \{s_0$: nothing, s_1: very low, s_2: low, s_3: medium, s_4: high, s_5: very high, s_6: perfect$\}$. Moreover, it is usually required that the linguistic term set satisfies the following additional characteristics.

1) There is a negation operator: $\text{Neg}(s_i) = s_{g-i}$, where $g + 1$ is the cardinality of the term set.
2) The set is ordered: $s_i \le s_j \Longleftrightarrow i \le j$. Therefore, there exist a maximization operator: $\max(s_i, s_j) = s_i$, if $s_j \le s_i$, and a minimization operator: $\min(s_i, s_j) = s_i$, if $s_i \le s_j$.

Definition 1 [20]: Let $S = \{s_0, s_1, \ldots, s_g\}$ be a linguistic term set. An HFLTS H_S on S is an ordered finite subset of consecutive linguistic terms in S.

Definition 2 [20]: Let $S = \{s_0, s_1, \ldots, s_g\}$ be a linguistic term set and H_S, H_S^1, and H_S^2 be three HFLTSs based on S.

1) The complement ${H_S}^c$ of H_S is defined by
$$ {H_S}^c = S - H_S = \{s_i \,|\, s_i \in S \text{ and } s_i \notin H_S\}. $$
2) The union $H_S^1 \cup H_S^2$ of H_S^1 and H_S^2 is defined by
$$ H_S^1 \cup H_S^2 = \{s_i \,|\, s_i \in H_S^1 \text{ or } s_i \in H_S^2\}. $$
3) The intersection $H_S^1 \cap H_S^2$ of H_S^1 and H_S^2 is defined by
$$ H_S^1 \cap H_S^2 = \{s_i \,|\, s_i \in H_S^1 \text{ and } s_i \in H_S^2\}. $$

We can easily see that the complement and the union that has been defined in Definition 2 are not closed on the set of all HFLTSs.

In order to compare two HFLTSs, Rodríguez *et al.* [20] introduced the definition of envelope for an HFLTS.

Definition 3 [20]: For an arbitrary HFLTS H_S, its upper bound ${H_S}^+$ and lower bound ${H_S}^-$ are defined as
$$ {H_S}^+ = \max\{s_i \,|\, s_i \in H_S\}, \quad {H_S}^- = \min\{s_i \,|\, s_i \in H_S\}. $$

Definition 4 [20]: The envelope, denoted by $\text{env}(H_S)$, of an HFLTS H_S, is a linguistic interval $[{H_S}^-, {H_S}^+]$, where ${H_S}^-$ and ${H_S}^+$ are the lower bound and the upper bound of H_S, respectively.

Using the envelope of an HFLTS, Rodríguez *et al.* [20] gave a method to compare two HFLTSs H_S^1 and H_S^2:
$$ H_S^1 > H_S^2 \text{ if and only if } \text{env}(H_S^1) > \text{env}(H_S^2) $$
$$ H_S^1 = H_S^2 \text{ if and only if } \text{env}(H_S^1) = \text{env}(H_S^2). $$

The comparisons between two linguistic intervals are the same as those of numerical intervals in [21] and [28].

As mentioned in the Introduction, if two HFLTSs have one common element, it is unreasonable to say one HFLTS is absolutely superior to another by the aforementioned method.

Example 1: Let $S = \{s_0$: nothing, s_1 : very low, s_2 : low, s_3 : medium, s_4 : high, s_5 : very high, s_6 : perfect$\}$ be a linguistic term set, $H_S^1 = \{s_2, s_3, s_4\}$, and $H_S^2 = \{s_4, s_5\}$ two HFLTSs on S.

From Definition 4, we have $\text{env}(H_S^1) = [s_2, s_4]$ and $\text{env}(H_S^2) = [s_4, s_5]$. According to the comparison between two numerical intervals that have been introduced by Wang *et al.* [28], the preference degree of $[s_4, s_5]$ over $[s_2, s_4]$ is
$$ p([s_4, s_5] > [s_2, s_4]) = \frac{\max(0, 5-2) - \max(0, 4-4)}{(5-4) + (4-2)} = 1. $$

Hence, $p(H_S^2 > H_S^1) = 1$; therefore, H_S^2 is absolutely superior to H_S^1. We know that the HFLTS $H_S^1 = \{s_2, s_3, s_4\}$ means that experts hesitate among linguistic terms s_2, s_3, and s_4 when they assess a linguistic variable, and $H_S^2 = \{s_4, s_5\}$ means that a linguistic variable may be s_4 or s_5. Compare H_S^1 and H_S^2. The linguistic term s_5 in H_S^2 is greater than any one in H_S^1, but s_4 is the possible linguistic term of a linguistic variable both for H_S^1 and H_S^2. Thus, we could not say that H_S^2 is absolutely superior to H_S^1. Since each HFLTS has finite linguistic terms, we think it is not suitable to compare them by the comparison method for numerical intervals.

In the following sections, we will define new operations with closed properties and give two new comparison methods.

Throughout the paper, let $\mathrm{Ind}(s_i)$ be the index i of a linguistic term s_i in a linguistic term set S, and let $\mathrm{Ind}(H_S)$ be the set of indexes of the linguistic terms in an HFLTS H_S on S.

III. Basic Operations on Hesitant Fuzzy Linguistic Term Sets

In [22] and [23], Torra defined the complement, union and intersection operations for hesitant fuzzy sets. In this section, we use Torra's idea to define the negation, max-union and min-intersection operations on HFLTSs.

Definition 5: Let $S = \{s_0, s_1, \ldots, s_g\}$ be a linguistic term set. For HFLTSs H_S, H_S^1, and H_S^2 on S,

1) we call $\{s_{g-i} \,|\, i \in \mathrm{Ind}(H_S)\}$ the negation of H_S, denoted by $\overline{H_S}$;
2) we call $\{\max\{s_i, s_j\} \,|\, s_i \in H_S^1, s_j \in H_S^2\}$ the max-union of H_S^1 and H_S^2, denoted by $H_S^1 \vee H_S^2$;
3) we call $\{\min\{s_i, s_j\} \,|\, s_i \in H_S^1, s_j \in H_S^2\}$ the min-intersection of H_S^1 and H_S^2, denoted by $H_S^1 \wedge H_S^2$.

In Example 1, $H_S^1 = \{s_2, s_3, s_4\}$ and $H_S^2 = \{s_4, s_5\}$. Then, by Definition 5, we have

$$\begin{aligned}\overline{(H_S^1)} &= \{s_{6-4}, s_{6-3}, s_{6-2}\} = \{s_2, s_3, s_4\}\\ H_S^1 \vee H_S^2 &= \{\max\{s_2, s_4\}, \max\{s_2, s_5\}, \max\{s_3, s_4\},\\ &\qquad \max\{s_3, s_5\}, \max\{s_4, s_4\}, \max\{s_4, s_5\}\}\\ &= \{s_4, s_5\}\end{aligned}$$

and

$$\begin{aligned}H_S^1 \wedge H_S^2 &= \{\min\{s_2, s_4\}, \min\{s_2, s_5\},\ \min\{s_3, s_4\}\\ &\qquad \min\{s_3, s_5\}, \min\{s_4, s_4\}, \min\{s_4, s_5\}\}\\ &= \{s_2, s_3, s_4\}.\end{aligned}$$

Remark 1: The results of the aforementioned operations are HFLTSs. In fact, for two HFLTSs, H_S^1 and H_S^2, assume that $H_S^{2+} \leq H_S^{1+}$. Then

$$\begin{aligned}&H_S^1 \vee H_S^2\\ &= \begin{cases} H_S^1, & H_S^{2-} \leq H_S^{1-}\\ \{s_i \,|\, i \in \{\mathrm{Ind}(H_S^{2-}), \mathrm{Ind}(H_S^{2-}) + 1 & \\ \quad \ldots, \mathrm{Ind}(H_S^{1+})\}\}, & H_S^{2-} > H_S^{1-}\end{cases}\\ &H_S^1 \wedge H_S^2\\ &= \begin{cases} H_S^2, & H_S^{2-} \leq H_S^{1-}\\ \{s_i \,|\, i \in \{\mathrm{Ind}(H_S^{1-}), \mathrm{Ind}(H_S^{1-}) + 1 & \\ \quad \ldots, \mathrm{Ind}(H_S^{2+})\}\}, & H_S^{2-} > H_S^{1-}.\end{cases}\end{aligned}$$

Property 1: Let $S = \{s_0, s_1, \ldots, s_g\}$ be a linguistic term set and H_S, H_S^1, H_S^2, and H_S^3 be four HFLTSs on S. Then, the following are true:

1) $\overline{(\overline{H_S})} = H_S$.
2) $\overline{(H_S^1 \vee H_S^2)} = \overline{(H_S^1)} \wedge \overline{(H_S^2)}$ and $\overline{(H_S^1 \wedge H_S^2)} = \overline{(H_S^1)} \vee \overline{(H_S^2)}$.
3) *Commutativity*: $H_S^1 \vee H_S^2 = H_S^2 \vee H_S^1$ and $H_S^1 \wedge H_S^2 = H_S^2 \wedge H_S^1$.
4) *Associativity*: $H_S^1 \vee (H_S^2 \vee H_S^3) = (H_S^1 \vee H_S^2) \vee H_S^3$ and $H_S^1 \wedge (H_S^2 \wedge H_S^3) = (H_S^1 \wedge H_S^2) \wedge H_S^3$.
5) *Distributivity*: $H_S^1 \wedge (H_S^2 \vee H_S^3) = (H_S^1 \wedge H_S^2) \vee (H_S^1 \wedge H_S^3)$ and $H_S^1 \vee (H_S^2 \wedge H_S^3) = (H_S^1 \vee H_S^2) \wedge (H_S^1 \vee H_S^3)$.

Proof: Let $\mathrm{Ind}(H_S)$ be a set of the indexes of all linguistic terms in H_S.

1) $\overline{(\overline{H_S})} = \{s_{g-j} \,|\, j \in \mathrm{Ind}(\overline{H_S})\} = \{s_{g-(g-i)} \,|\, i \in \mathrm{Ind}(H_S)\} = \{s_i \,|\, i \in \mathrm{Ind}(H_S)\} = H_S$.
2) According to (2) in Definition 5, we have

$$\begin{aligned}H_S^1 \vee H_S^2 &= \{\max\{s_i, s_j\} \,|\, s_i \in H_S^1, s_j \in H_S^2\}\\ &= \{s_{\max\{i,j\}} \,|\, i \in \mathrm{Ind}(H_S^1), j \in \mathrm{Ind}(H_S^2)\}.\end{aligned}$$

Thus

$$\overline{(H_S^1 \vee H_S^2)} = \{s_{g-\max\{i,j\}} \,|\, i \in \mathrm{Ind}(H_S^1), j \in \mathrm{Ind}(H_S^2)\}.$$

On the other hand, we obtain

$$\begin{aligned}&\overline{(H_S^1)} \wedge \overline{(H_S^2)} = \{s_{g-i} \,|\, i \in \mathrm{Ind}(H_S^1)\} \wedge \{s_{g-j} \,|\, j \in \mathrm{Ind}(H_S^2)\}\\ &\quad = \{\min\{s_{g-i}, s_{g-j}\} \,|\, i \in \mathrm{Ind}(H_S^1), j \in \mathrm{Ind}(H_S^2)\}\\ &\quad = \{s_{g-\max\{i,j\}} \,|\, i \in \mathrm{Ind}(H_S^1), j \in \mathrm{Ind}(H_S^2)\}.\end{aligned}$$

Therefore, $\overline{(H_S^1 \vee H_S^2)} = \overline{(H_S^1)} \wedge \overline{(H_S^2)}$.

Similarly, the other equation can be proved.

3) It is a direct result of Definition 5.
4) From (2) in Definition 5, we have

$$H_S^2 \vee H_S^3 = \{s_{\max\{j,k\}} \,|\, j \in \mathrm{Ind}(H_S^2), k \in \mathrm{Ind}(H_S^3)\}.$$

Thus

$$\begin{aligned}&H_S^1 \vee (H_S^2 \vee H_S^3)\\ &= \{\max\{s_i, s_{\max\{j,k\}}\} \,|\, s_i \in H_S^1, j \in \mathrm{Ind}(H_S^2), k \in \mathrm{Ind}(H_S^3)\}\\ &= \{s_{\max\{i,j,k\}} \,|\, i \in \mathrm{Ind}(H_S^1), j \in \mathrm{Ind}(H_S^2), k \in \mathrm{Ind}(H_S^3)\}.\end{aligned}$$

Since

$$\begin{aligned}&(H_S^1 \vee H_S^2) \vee H_S^3\\ &= \{\max\{s_{\max\{i,j\}}, s_k\} \,|\, i \in \mathrm{Ind}(H_S^1), j \in \mathrm{Ind}(H_S^2), s_k \in H_S^3\}\\ &= \{s_{\max\{i,j,k\}} \,|\, i \in \mathrm{Ind}(H_S^1), j \in \mathrm{Ind}(H_S^2), k \in \mathrm{Ind}(H_S^3)\}\end{aligned}$$

we get $H_S^1 \vee (H_S^2 \vee H_S^3) = (H_S^1 \vee H_S^2) \vee H_S^3$.

Similarly, we can prove the equality for the min-intersection operation.

5) From

$$H_S^2 \vee H_S^3 = \{s_{\max\{j,k\}} \,|\, j \in \mathrm{Ind}(H_S^2), k \in \mathrm{Ind}(H_S^3)\}$$

we have

$$\begin{aligned}&H_S^1 \wedge (H_S^2 \vee H_S^3)\\ &= \{\min\{s_i, s_{\max\{j,k\}}\} \,|\, s_i \in H_S^1, j \in \mathrm{Ind}(H_S^2), k \in \mathrm{Ind}(H_S^3)\}\\ &= \{s_{\min\{i,\max\{j,k\}\}} \,|\, i \in \mathrm{Ind}(H_S^1), j \in \mathrm{Ind}(H_S^2), k \in \mathrm{Ind}(H_S^3)\}\\ &= \{s_{\max\{\min\{i,j\},\min\{i,k\}\}} \,|\, i \in \mathrm{Ind}(H_S^1), j \in \mathrm{Ind}(H_S^2)\\ &\qquad\qquad k \in \mathrm{Ind}(H_S^3)\}.\end{aligned}$$

In addition, from

$$\begin{aligned}H_S^1 \wedge H_S^2 &= \{\min\{s_{i_1}, s_j\} \,|\, s_{i_1} \in H_S^1, s_j \in H_S^2\}\\ &= \{s_{\min\{i_1,j\}} \,|\, i_1 \in \mathrm{Ind}(H_S^1), j \in \mathrm{Ind}(H_S^2)\}\end{aligned}$$

and

$$H_S^1 \wedge H_S^3 = \{\min\{s_{i_2}, s_k\} \,|\, s_{i_2} \in H_S^1, s_k \in H_S^3\}$$
$$= \{s_{\min\{i_2,k\}} \,|\, i_2 \in \mathrm{Ind}(H_S^1), k \in \mathrm{Ind}(H_S^3)\}$$

we have

$$\begin{aligned}(H_S^1 \wedge H_S^2) \vee (H_S^1 \wedge H_S^3) &= \{\max\{s_{\min\{i_1,j\}}, s_{\min\{i_2,k\}}\} \,|\, i_1, i_2 \in \mathrm{Ind}(H_S^1)\\ &\qquad j \in \mathrm{Ind}(H_S^2), k \in \mathrm{Ind}(H_S^3)\}\\ &= \{s_{\max\{\min\{i_1,j\},\min\{i_2,k\}\}} \,|\, i_1, i_2 \in \mathrm{Ind}(H_S^1)\\ &\qquad j \in \mathrm{Ind}(H_S^2), k \in \mathrm{Ind}(H_S^3)\}.\end{aligned}$$

Thus, $H_S^1 \wedge (H_S^2 \vee H_S^3) = (H_S^1 \wedge H_S^2) \vee (H_S^1 \wedge H_S^3)$.

We can also get the other equality in a similar way.

IV. Possibility Degree Formula for Ranking Hesitant Fuzzy Linguistic Term Sets

The theory of HFLTSs' comparison is very important. Making use of the theory, one can rank alternatives or select the best alternative. In [20], Rodríguez *et al.* used the comparison theory of interval values to rank HFLTSs. In this section, we will give some comparison methods of HFLTSs, which are based on the probability theory.

In order to introduce a possibility degree formula for ranking two HFLTSs, we first use an example to illustrate the main idea of our method. Let $S = \{s_0$: nothing, s_1: very low, s_2: low, s_3: medium, s_4: high, s_5: very high, s_6: perfect$\}$ be a linguistic term set, and $H_S^1 = \{s_3, s_4, s_5, s_6\}$ and $H_S^2 = \{s_2, s_3, s_4\}$ be two HFLTSs on S. Clearly, H_S^1 and H_S^2 have the common linguistic terms s_3 and s_4. We write them as the following forms:

$$H_S^1: \quad s_3, s_4, s_5, s_6$$
$$H_S^2: \quad s_2, s_3, s_4.$$

We add one linguistic term $\bar{s}_2$ in H_S^1 and two linguistic terms $\bar{s}_5$ and $\bar{s}_6$ in H_S^2, where $\bar{s}_2$ can be any linguistic term in H_S^1, and $\bar{s}_5, \bar{s}_6$ can be any linguistic terms in H_S^2. Then, we obtain two new linguistic term sets, denoted by H_1^* and H_2^*, as follows:

$$H_1^*: \quad \bar{s}_2, s_3, s_4, s_5, s_6$$
$$H_2^*: \quad s_2, s_3, s_4, \bar{s}_5, \bar{s}_6.$$

We note that the way to construct H_i^* by adding linguistic terms in H_S^i can keep the meaning represented by H_S^i unchanged. Therfore, in order to compare H_S^1 and H_S^2, we only need to compare H_1^* and H_2^*. Now, compare the linguistic terms in the corresponding place in H_1^* and H_2^*. We note that H_1^* has three linguistic terms greater than the corresponding ones in H_2^*: $\bar{s}_2 > s_2, s_5 > \bar{s}_5$ and $s_6 > \bar{s}_6$, and H_1^* and H_2^* have two same linguistic terms, s_3 and s_4, in their corresponding places. There are five different linguistic terms, s_2, s_3, s_4, s_5, and s_6, in H_1^* and H_2^*. Thus, we regard the ratio $\frac{2\times 0.5+3}{5} = 0.8$ as the possibility degree of H_S^1 being not less than H_S^2.

For a general case, let $S = \{s_0, s_1, \ldots, s_g\}$ be a linguistic term set, and H_S^1 and H_S^2 be two HFLTSs on S. In a similar way to the above, we can construct two linguistic term sets H_1^* and H_2^*. Let $H_{S(1,2)}^* = \{s_i \,|\, s_i \in H_S^1 \text{and } s_i \in H_S^2\}$ be the set of the common linguistic terms in H_S^1 and H_S^2, and let $H_{H_1^* > H_2^*} = \{s_i^1 \,|\, s_i^1 \in H_1^*, s_i^2 \in H_2^*, s_i^1 > s_i^2\}$ be the set of all linguistic terms in H_1^* larger than the corresponding terms in H_2^*. For a set X, we let $|X|$ be its cardinal number.

Definition 6: We call the ratio $\frac{0.5|H_{S(1,2)}^*| + |H_{H_1^* > H_2^*}|}{|H_1^*|}$ the possibility degree of H_S^1 being not less than H_S^2, denoted by $p(H_S^1 \geq H_S^2)$.

From the possible position relationships of two HFLTSs, we can give a concrete formula for the possibility degree. For HFLTSs H_S^1 and H_S^2 on S, let H_S^{i-} and H_S^{i+} be the lower bound and the upper bound of H_S^i, respectively, for $i = 1, 2$. Suppose that $\mathrm{Ind}(H_S^{1-}) = i_1, \mathrm{Ind}(H_S^{1+}) = i_m, \mathrm{Ind}(H_S^{2-}) = j_1$ and $\mathrm{Ind}(H_S^{2+}) = j_n$. If $H_S^{1+} \leq H_S^{2+}$, that is, $i_m \leq j_n$, then the possibility degree of $H_S^1 \geq H_S^2 p(H_S^1 \geq H_S^2)$ is

$$p(H_S^1 \geq H_S^2) = \begin{cases} 0, & i_m < j_1 \\ \dfrac{0.5(i_m - j_1 + 1)}{j_n - i_1 + 1}, & i_1 \leq j_1 \leq i_m \leq j_n \\ \dfrac{i_1 - j_1 + 0.5(i_m - i_1 + 1)}{j_n - j_1 + 1}, & j_1 < i_1 \leq i_m < j_n. \end{cases} \tag{1}$$

If $H_S^{1+} > H_S^{2+}$, i.e., $i_m > j_n$, then

$$p(H_S^1 \geq H_S^2) = \begin{cases} 1, & j_n < i_1 \\ \dfrac{0.5(j_n - i_1 + 1) + (i_m - j_n) + (i_1 - j_1)}{i_m - j_1 + 1}, & j_1 \leq i_1 \leq j_n \leq i_m \\ \dfrac{0.5(j_n - j_1 + 1) + (i_m - j_n)}{i_m - i_1 + 1}, & i_1 < j_1 \leq j_n < i_m. \end{cases} \tag{2}$$

We note that if $j_n < i_1$ or $i_m < j_1$, the two HFLTSs H_S^1 and H_S^2 have no common elements; in this case, we may use $p(H_S^1 > H_S^2)$ to denote the possibility degree of H_S^1 greater than H_S^2. Then, $p(H_S^1 > H_S^2) = 0$ or 1.

Remark 2: Suppose $H_S^1 = \{s_i\}$ and $H_S^2 = \{s_j\}$. Then, from Definition 6

$$p(H_S^1 \geq H_S^2) = \begin{cases} 1, & s_i > s_j \\ \dfrac{1}{2}, & s_i = s_j \\ 0, & s_i < s_j. \end{cases}$$

Property 2 (Complementarity): $p(H_S^1 \geq H_S^2) + p(H_S^2 \geq H_S^1) = 1$; especially, if $H_S^1 = H_S^2$, then $p(H_S^1 \geq H_S^2) = p(H_S^2 \geq H_S^1) = 0.5$.

Proof: Suppose that $\mathrm{Ind}(H_S^{1-}) = i_1, \mathrm{Ind}(H_S^{1+}) = i_m$, $\mathrm{Ind}(H_S^{2-}) = j_1, \mathrm{Ind}(H_S^{2+}) = j_n, i_m \le j_n$. Then, by Definition 6

$$p(H_S^1 \ge H_S^2) = \begin{cases} 0, & i_m < j_1 \\ \dfrac{0.5(i_m - j_1 + 1)}{j_n - i_1 + 1}, & i_1 \le j_1 \le i_m \le j_n \\ \dfrac{i_1 - j_1 + 0.5(i_m - i_1 + 1)}{j_n - j_1 + 1}, & j_1 < i_1 \le i_m < j_n \end{cases} \quad (3)$$

and

$$p(H_S^2 \ge H_S^1) = \begin{cases} 1, & i_m < j_1 \\ \dfrac{j_1 - i_1 + 0.5(i_m - j_1 + 1) + j_n - i_m}{j_n - i_1 + 1}, & i_1 \le j_1 \le i_m \le j_n \\ \dfrac{0.5(i_m - i_1 + 1) + j_n - i_m}{j_n - j_1 + 1}, & j_1 < i_1 \le i_m < j_n. \end{cases} \quad (4)$$

Thus, $p(H_S^1 \ge H_S^2) + p(H_S^2 \ge H_S^1) = 1$.

By the aforementioned property, we give the following definition.

Definition 7: If $p(H_S^1 \ge H_S^2) > p(H_S^2 \ge H_S^1)$, then we say that H_S^1 is superior to H_S^2 with the degree of $p(H_S^1 \ge H_S^2)$, denoted by $H_S^1 \succ^{p(H_S^1 \ge H_S^2)} H_S^2$. In this case, we also say that H_S^2 is inferior to H_S^1 with the degree of $p(H_S^1 \ge H_S^2)$, denoted by $H_S^2 \prec^{p(H_S^1 \ge H_S^2)} H_S^1$.

If $p(H_S^1 \ge H_S^2) = 1$, then we say that H_S^1 is absolutely superior to H_S^2, or H_S^2 is absolutely inferior to H_S^1.

If $p(H_S^1 \ge H_S^2) = 0.5$, then we say that H_S^1 is indifferent with H_S^2, denoted by $H_S^1 \sim H_S^2$.

From Formula (2), we can see that H_S^1 is absolutely superior to H_S^2 if and only if $\mathrm{Ind}(H_S^{1-}) > \mathrm{Ind}(H_S^{2+})$. In Example 1, $H_S^1 = \{s_2, s_3, s_4\}$ and $H_S^2 = \{s_4, s_5\}$. Then, by Formula (2), we can obtain $p(H_S^2 \ge H_S^1) = \frac{(4-2)+0.5+(5-4)}{4} = 0.875$. The comparison result implies H_S^2 is not absolutely superior to H_S^1 and consistent with our analysis in Section II.

Property 3: Suppose that $\mathrm{Ind}(H_S^{1-}) = i_1, \mathrm{Ind}(H_S^{1+}) = i_m, \mathrm{Ind}(H_S^{2-}) = j_1$, and $\mathrm{Ind}(H_S^{2+}) = j_n$. Then

1) $p(H_S^1 \ge H_S^2) < 0.5$ if and only if $i_1 + i_m < j_1 + j_n$;
2) $p(H_S^1 \ge H_S^2) = 0.5$ if and only if $i_1 + i_m = j_1 + j_n$;
3) $p(H_S^1 \ge H_S^2) > 0.5$ if and only if $i_1 + i_m > j_1 + j_n$.

Proof: Since the proof of (1), (2), and (3) is similar, we only give the proof of (1). From Formulas (1) and (2), we can calculate the possibility degree $p(H_S^1 \ge H_S^2)$ by two separate cases: $H_S^{1+} \le H_S^{2+}$ and $H_S^{1+} > H_S^{2+}$.

Suppose $H_S^{1+} \le H_S^{2+}$, i.e., $i_m \le j_n$. Then, by the Formula (1), we have that, $p(H_S^1 \ge H_S^2) < 0.5$ if and only if, $i_m < j_1$, or $\frac{0.5(i_m - j_1 + 1)}{j_n - i_1 + 1} < 0.5$, for $i_1 \le j_1 \le i_m \le j_n$, or $\frac{i_1 - j_1 + 0.5(i_m - i_1 + 1)}{j_n - j_1 + 1} < 0.5$, for $j_1 < i_1 \le i_m < j_n$, if and only if, $i_m < j_1$, or $i_1 + i_m < j_1 + j_n$, for $i_1 \le j_1 \le i_m \le j_n$, or $i_1 + i_m < j_1 + j_n$, for $j_1 < i_1 \le i_m < j_n$, if and only if, $i_1 + i_m < j_1 + j_n$.

Suppose $H_S^{1+} > H_S^{2+}$, i.e., $i_m > j_n$. In this case, if $j_1 \le i_1 \le j_n \le i_m$, then $p(H_S^1 \ge H_S^2) = \frac{0.5(j_n - i_1 + 1) + (i_m - j_n) + (i_1 - j_1)}{i_m - j_1 + 1} = 0.5 + 0.5\frac{i_1 + i_m - j_1 - j_n}{i_m - j_1 + 1} \ge 0.5$. Hence, by (2), $p(H_S^1 > H_S^2) < 0.5$, if and only if, $\frac{0.5(j_n - j_1 + 1) + (i_m - j_n)}{i_m - i_1 + 1} < 0.5$, for $i_1 < j_1 \le j_n < i_m$, if and only if, $i_1 + i_m < j_1 + j_n$ for $i_1 < j_1 \le j_n < i_m$, if and only if $i_1 + i_m < j_1 + j_n$.

The following transitivity can be derived from Property 3.

Property 4 (Transitivity): Let $S = \{s_0, s_1, \ldots, s_g\}$ be a linguistic term set, H_S^1, H_S^2, and H_S^3 be three HFLTSs on S.

If $p(H_S^1 \ge H_S^2) > 0.5$ and $p(H_S^2 \ge H_S^3) \ge 0.5$, or $p(H_S^1 \ge H_S^2) \ge 0.5$ and $p(H_S^2 \ge H_S^3) > 0.5$, then $p(H_S^1 \ge H_S^3) > 0.5$.

If $p(H_S^1 \ge H_S^2) = 0.5$ and $p(H_S^2 \ge H_S^3) = 0.5$, then $p(H_S^1 \ge H_S^3) = 0.5$.

Fan and Liu [8] proposed a method to compare two ordinal interval numbers. We find that the rationale of Fan and Liu's method can be used to compare HFLTSs. For two HFLTSs H_S^1 and H_S^2, let $s_i \in H_S^1$ and $s_j \in H_S^2$. Suppose that s_i and s_j are uniformly and independently distributed in H_S^1 and H_S^2, respectively. The possibility of $s_i > s_j, s_i < s_j$, and $s_i = s_j$ are denoted as $p_{s_i > s_j}, p_{s_i < s_j}$, and $p_{s_i = s_j}$, respectively. Then, from the rationale of Fan and Liu's method, $\sum_{s_i \in H_S^1, s_j \in H_S^2} (p_{s_i > s_j} + 0.5 p_{s_i = s_j})$ is called the possibility degree of H_S^1 being not less than H_S^2, denoted by $p_F(H_S^1 \ge H_S^2)$.

From the three possible position relationships of two HFLTSs, we can obtain the following formulas:

If $H_S^{1+} \le H_S^{2+}$, that is, $i_m \le j_n$, then

$$p_F(H_S^1 \ge H_S^2) = \begin{cases} 0, & i_m < j_1 \\ 0.5\left(\dfrac{i_m - j_1 + 1}{i_m - i_1 + 1}\right)\left(\dfrac{i_m - j_1 + 1}{j_n - j_1 + 1}\right), & i_1 \le j_1 \le i_m \le j_n \\ \dfrac{i_1 - j_1}{j_n - j_1 + 1} + 0.5\dfrac{i_m - i_1 + 1}{j_n - j_1 + 1}, & j_1 < i_1 \le i_m < j_n. \end{cases} \quad (5)$$

If $H_S^{1+} > H_S^{2+}$, that is, $i_m > j_n$, then

$$p_F(H_S^1 \ge H_S^2) = \begin{cases} 1, & j_n < i_1 \\ \dfrac{i_m - j_n}{i_m - i_1 + 1} + \dfrac{j_n - i_1 + 1}{i_m - i_1 + 1} \left(0.5\dfrac{j_n - i_1 + 1}{j_n - j_1 + 1} + \dfrac{i_1 - j_1}{j_n - j_1 + 1}\right), & j_1 \le i_1 \le j_n \le i_m \\ \dfrac{i_m - j_n}{i_m - i_1 + 1} + 0.5\left(\dfrac{j_n - j_1 + 1}{i_m - i_1 + 1}\right), & i_1 < j_1 \le j_n < i_m. \end{cases} \quad (6)$$

The possibility degree $p_F(H_S^1 \ge H_S^2)$ also satisfies the above three properties. Comparing (5) and (6) with (1) and (2), we can see that $p_F(H_S^1 \ge H_S^2) = p(H_S^1 \ge H_S^2)$ except the overlapping case: $i_1 \le j_1 \le i_m \le j_n$ or $j_1 \le i_1 \le j_n \le i_m$.

Example 2: Let $S = \{s_0$: nothing, s_1: very low, s_2: low, s_3: medium, s_4: high, s_5: very high, s_6: perfect$\}$ be a linguistic term set and $H_S^1 = \{s_3, s_4, s_5\}, H_S^2 = \{s_4, s_5, s_6\}, H_S^3 = \{s_5\}, H_S^4 = \{s_1, s_2, s_3\}$, and $H_S^5 = \{s_3, s_4\}$ be five HFLTSs. Using Formula (1) or (2) and (5) or (6), we obtain:

$p_F(H_S^2 \geq H_S^1) = \frac{1}{3} + \frac{1}{3} \times \frac{2}{3} + 0.5 \times \frac{2}{3} \times \frac{2}{3} \approx 0.778$.
$p(H_S^2 \geq H_S^1) = \frac{2+0.5\times 2}{4} = 0.75$.
$p_F(H_S^5 \geq H_S^1) = 0.5 \times \frac{2}{3} \approx 0.333$.
$p(H_S^5 \geq H_S^1) = \frac{0.5\times 2}{3} \approx 0.333$.
$p_F(H_S^4 \geq H_S^5) = 0.5 \times \frac{1}{3} \times \frac{1}{2} \approx 0.083$.
$p(H_S^4 \geq H_S^5) = \frac{0.5\times 1}{4} = 0.125$.

Now, we study a method for ranking HFLTSs. Clearly, the possibility degree formulas (1) and (2) or (5) and (6) can be used to compare two HFLTSs. For n HFLTSs, we need a similar argument to possibility degree method in [27]; therefore, we refer to the Appendix of this paper. We may rank the five HFLTSs in Example 2 to illustrate the application of the possibility degree method in Appendix.

First, by Step 1 and Step 2 in the Appendix and (1) or (2), we construct the possibility degree matrix P and the preference relation matrix U

$$P = \begin{pmatrix} 0.5000 & 0.2500 & 0.1667 & 0.9000 & 0.6667 \\ 0.7500 & 0.5000 & 0.5000 & 1.0000 & 0.8750 \\ 0.8333 & 0.5000 & 0.5000 & 1.0000 & 1.0000 \\ 0.1000 & 0.0000 & 0.0000 & 0.5000 & 0.1250 \\ 0.3333 & 0.1250 & 0.0000 & 0.8750 & 0.5000 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

By Step 3 in the Appendix, we get $V_1 = \{H_S^2, H_S^3\}, V_2 = \{H_S^1\}, V_3 = \{H_S^5\}$, and $V_4 = \{H_S^4\}$. Since $\mathrm{Ind}(H_S^{3+}) - \mathrm{Ind}(H_S^{3-}) < \mathrm{Ind}(H_S^{2+}) - \mathrm{Ind}(H_S^{2-})$, we get H_S^3 is quasi-superior to H_S^2 by Step 4. Thus, the ranking result of the HFLTSs is $H_S^3 \rhd H_S^2 \succ^{0.750} H_S^1 \succ^{0.667} H_S^5 \succ^{0.875} H_S^4$. If the possibility degrees are calculated by (5) or (6), then the ranking result is $H_S^3 \rhd H_S^2 \succ^{0.778} H_S^1 \succ^{0.667} H_S^5 \succ^{0.917} H_S^4$. By using our method and Fan and Liu's method to compare n HFLTSs, the ranking orders of HFLTSs are the same, but the possibility degrees are not the same for the overlapping case.

V. Two Hesitant Fuzzy Linguistic Operators and Their Applications in Decision Making

In this section, we generalize the LWA and LOWA operators to HFLTS context, and define a hesitant fuzzy LWA (HLWA) operator and a hesitant fuzzy LOWA (HLOWA) operator. Then, we apply these two operators to deal with MCDM problems with linguistic information modeled by HFLTSs.

A. Convex Combination Operation and Two Aggregation Operators

We first recall the definition of the convex combination of two linguistic terms, given in [7]. Let $S = \{s_0, s_1, \ldots, s_g\}$ be a linguistic term set. For two linguistic terms s_i and s_j in S, the convex combination of s_i and s_j is defined as

$$C^2(w_1, s_i, w_2, s_j) = w_1 \odot s_i \oplus w_2 \odot s_j = s_k$$

where $w_i \geq 0 (i = 1, 2), w_1 + w_2 = 1, k = \min\{g, \mathrm{round}((w_1)i + (1 - w_1)j)\}$, and "round" is the usual round operation.

Using the convex combination of linguistic terms, we introduce a convex combination of two HFLTSs.

Definition 8: Let $S = \{s_0, s_1, \ldots, s_g\}$ be a linguistic term set and H_S^1 and H_S^2 be two HFLTSs on S. A convex combination of H_S^1 and H_S^2 is defined as

$$C^2(w_1, H_S^1, w_2, H_S^2) = w_1 \odot H_S^1 \oplus w_2 \odot H_S^2 = \{C^2(w_1, a_1, w_2, a_2) | a_1 \in H_S^1, a_2 \in H_S^2\}$$

where $w_i \geq 0 (i = 1, 2)$ and $w_1 + w_2 = 1$.

We now prove that the convex combination of two HFLTSs is also an HFLTS. The following lemma is an easy fact; therefore, we omit its proof.

Lemma 1: Let $x_i, i = 1, 2, \ldots, n$, be real numbers with $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$. Suppose $x_i \leq x_{i+1} \leq x_i + 1$ for $1 \leq i \leq n - 1$. Then, $\{\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n\} = \{k \in \mathbb{Z} | \bar{x}_1 \leq k \leq \bar{x}_n\}$, where $\mathbb{Z}$ is the set of all integers, $\bar{x}_i = \mathrm{round}(x_i)$ for $1 \leq i \leq n$.

Property 5: Let $S = \{s_0, s_1, \ldots, s_g\}$ be a linguistic term set and $H_S^1 = \{s_i, s_{i+1}, \ldots, s_{i+n}\}$ and $H_S^2 = \{s_j, s_{j+1}, \ldots, s_{j+m}\}$ be two HFLTSs on S. For $0 \leq \lambda \leq 1$, let $a_{rs} = \lambda(i + r) + (1 - \lambda)(j + s)$ and $\bar{a}_{rs} = \mathrm{round}(a_{rs})$ for $0 \leq r \leq n$ and $0 \leq s \leq m$. Then, the convex combination $C^2\{\lambda, H_S^1, 1 - \lambda, H_S^2\}$ of H_S^1 and H_S^2 is also an HFLTS and equal to $\{s_k | k \in \mathbb{Z}, \bar{a}_{00} \leq k \leq \bar{a}_{nm}\}$.

Proof: By the hypothesis, we have the following inequalities:

$$0 \leq a_{00} \leq a_{rs} \leq a_{nm} \text{ for } 0 \leq r \leq n \text{ and } 0 \leq s \leq m$$

$$0 \leq a_{00} \leq a_{01} \leq a_{02} \leq \cdots \leq a_{0m} \leq a_{1m} \leq a_{2m} \leq \cdots \leq a_{nm}, \text{and}$$

$$a_{0s} \leq a_{0s+1} \leq a_{0s} + (1 - \lambda) \leq a_{0s} + 1 \text{ for } 0 \leq s \leq m - 1$$

$$a_{rm} \leq a_{r+1m} \leq a_{rm} + \lambda \leq a_{rm} + 1 \text{ for } 0 \leq r \leq n - 1.$$

By Lemma 1, we have $\{\bar{a}_{00}, \bar{a}_{01}, \ldots, \bar{a}_{0m}, \bar{a}_{1m}, \ldots, \bar{a}_{nm}\} = \{k \in \mathbb{Z} | \bar{a}_{00} \leq k \leq \bar{a}_{nm}\}$. Hence, $C^2\{\lambda, H_S^1, 1 - \lambda, H_S^2\} = \{\bar{a}_{rs} | 0 \leq r \leq n, 0 \leq s \leq m\} = \{k \in \mathbb{Z} | \bar{a}_{00} \leq k \leq \bar{a}_{nm}\}$.

Based on convex combinations of two HFLTSs, we define the following hesitant fuzzy linguistic operators.

Definition 9: Let $S = \{s_0, s_1, \ldots, s_g\}$ be a linguistic term set, $H_S^1, H_S^2, \ldots, H_S^n$ be n HFLTSs on S. Let $w = (w_1, w_2, \ldots, w_n)^T$ be a weighting vector of $H_S^j (j = 1, 2, \ldots, n)$ with $w_j \geq 0 (i = 1, 2, \ldots, n)$ and $\sum_{j=1}^n w_j = 1$. Then, the hesitant fuzzy linguistic WA (HLWA) operator is defined as

$$\mathrm{HLWA}(H_S^1, H_S^2, \ldots, H_S^n) = C^n\{w_k, H_S^k, k = 1, \ldots, n\} = w_1 \odot H_S^1 \oplus (1 - w_1) \odot C^{n-1}\left\{ w_h \Big/ \sum_{k=2}^n w_k, H_S^h, h = 2, \ldots, n \right\}.$$

Definition 10: Let $S, H_S^i (i = 1, 2, \ldots, n)$ be as in Definition 9. The hesitant fuzzy LOWA (HLOWA) operator is defined as

$$\begin{aligned} &\text{HLOWA}(H_S^1, H_S^2, \ldots, H_S^n) \\ &\quad = C^n\{w_k, H_S^{\sigma_k}, k = 1, 2, \ldots, n\} = w_1 \odot H_S^{\sigma_1} \oplus (1 - w_1) \\ &\quad \odot C^{n-1}\left\{ w_h \Big/ \sum_{k=2}^{n} w_k, H_S^{\sigma_h}, h = 2, 3, \ldots, n \right\} \end{aligned}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is an associated weighting vector of the operator with $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$; $(H_S^{\sigma_1}, H_S^{\sigma_2}, \ldots, H_S^{\sigma_n})$ is a permutation of $(H_S^1, H_S^2, \ldots, H_S^n)$ such that $H_S^{\sigma_i} \succ H_S^{\sigma_j}$ or $H_S^{\sigma_i} \rhd H_S^{\sigma_j}$ for all $i < j$.

Many approaches have been developed for determining the associated weighting vector $w = (w_1, w_2, \ldots, w_n)^T$ of the OWA operator, which were made a detailed overview in [33]. Different methods reflect different attitudes of a decision maker or his/her requirements for aggregated arguments. These approaches are effective for determining the weighting vector, which are associated to the HLOWA operator.

Example 3: Let $S = \{s_0\text{: nothing}, s_1\text{: very low}, s_2\text{: low}, s_3\text{: medium}, s_4\text{: high}, s_5\text{: very high}, s_6\text{: perfect}\}$ be a linguistic term set and $H_S^1 = \{s_2, s_3, s_4\}, H_S^2 = \{s_4, s_5\}$, and $H_S^3 = \{s_3\}$ be three HFLTSs on S. Let $w = (0.25, 0.5, 0.25)^T$ be the associated weighting vector.

In order to aggregate the three HFLTSs, we first use the possibility degree method in Appendix to rank them.

By Step 1 and Step 2, we obtain the possibility degree matrix P and the preference relation matrix U

$$P = \begin{pmatrix} 0.500 & 0.125 & 0.500 \\ 0.875 & 0.5000 & 1.000 \\ 0.500 & 0.000 & 0.500 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

By Step 3 and Step 4, we have $H_S^2 \succ^{0.875} H_S^3 \rhd H_S^1$. Then, the aggregation value given by the HLOWA operator with $w = (0.25, 0.5, 0.25)^T$ is as follows:

$$\begin{aligned} &\text{HLOWA}(H_S^1, H_S^2, H_S^3) \\ &\quad = 0.25 \odot H_S^2 \oplus 0.75 \odot C^2 \left\{ \frac{2}{3}, H_S^3, \frac{1}{3}, H_S^1 \right\} \\ &\quad = 0.25 \odot \{s_4, s_5\} \oplus 0.75 \odot \{s_3\} = \{s_3, s_4\}. \end{aligned}$$

The aggregation results of HFLTSs by the two aforementioned operators are HFLTSs. We list some properties of the two operators and omit their proof.

Property 6: Let $S = \{s_0, s_1, \ldots, s_g\}$ be a linguistic term set and $(H_S^1, H_S^2, \ldots, H_S^n)$ be a collection of HFLTSs on S. Then, the HLWA and HLOWA operators satisfy the following properties:

1) (Boundary) If there don't exist indifferent elelents among the n HFLTSs, then

$$\max_i \{H_S^i\} \succ \text{HLWA}(H_S^1, H_S^2, \ldots, H_S^n) \succ \min_i \{H_S^i\}$$

$$\max_i \{H_S^i\} \succ \text{HLOWA}(H_S^1, H_S^2, \ldots, H_S^n) \succ \min_i \{H_S^i\}$$

where $\max_i \{H_S^i\}$ and $\min_i \{H_S^i\}$ are the most superior element and the most inferior element among the n HFLTSs, respectively.

2) (Monotonicity) For two ordered collections $(H_S^{\alpha_1}, H_S^{\alpha_2}, \ldots, H_S^{\alpha_n})$ and $(H_S^{\beta_1}, H_S^{\beta_2}, \ldots, H_S^{\beta_n})$ of HFLTSs, with $H_S^{\alpha_i} > H_S^{\beta_i}$ for all i, we have

$$\text{HLWA}(H_S^{\alpha_1}, \ldots, H_S^{\alpha_n}) > \text{HLWA}(H_S^{\beta_1}, \ldots, H_S^{\beta_n})$$

$$\text{HLOWA}(H_S^{\alpha_1}, \ldots, H_S^{\alpha_n}) > \text{HLOWA}(H_S^{\beta_1}, \ldots, H_S^{\beta_n}).$$

3) (Commutativity) If $(H_S^{\beta_1}, H_S^{\beta_2}, \ldots, H_S^{\beta_n})$ is a permutation of $(H_S^1, H_S^2, \ldots, H_S^n)$, then

$$\text{HLOWA}(H_S^{\beta_1}, \ldots, H_S^{\beta_n}) = \text{HLOWA}(H_S^1, \ldots, H_S^n).$$

4) (Idempotency): If $H_S^j = H_S$ for all j, then $\text{HLWA}(H_S^1, \ldots, H_S^n) = \text{HLOWA}(H_S^1, \ldots, H_S^n) = H_S$.

B. Multicriteria Decision Making Method Based the HLWA and HLOWA Operators

A MCDM problem considered in this paper can be described as follows: let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of alternatives, $C = \{c_1, c_2, \ldots, c_m\}$ be a set of criteria, and $S = \{s_0, s_1, \ldots, s_g\}$ be a linguistic term set. Based on the linguistic term set S, an expert provides his/her evaluations about alternatives $x_i (i = 1, 2, \ldots, n)$ under criteria $c_j (j = 1, 2, \ldots, m)$ by using linguistic expressions, $ll(x_i, c_j)$, which can be transformed into HFLTSs H_S^{ij} (see [20]). A decision-maker's goal is to obtain the ranking order of the alternatives.

Applying the HLOWA and HLWA operators on HFLTSs and the possibility degree method in the Appendix, we introduce a ranking method of the alternatives by the following steps.

Step 1: If the importance weights of criteria are unknown, then we utilize the HLOWA operator to derive the overall aggregation values H_S^i of alternatives x_i for $i = 1, 2, \ldots, n$

$$H_S^i = \text{HLOWA}(H_S^{i1}, H_S^{i2}, \ldots, H_S^{im}).$$

If an importance weighting vector, $p = (p_1, p_2, \ldots, p_m)^T$ with $p_j \geq 0$ and $\sum_{j=1}^m p_j = 1$, of criteria is given, then we utilize the HLWA operator to derive the overall aggregation values H_S^i of alternatives x_i for $i = 1, 2, \ldots, n$

$$H_S^i = \text{HLWA}(H_S^{i1}, H_S^{i2}, \ldots, H_S^{im}).$$

Step 2: Compare $H_S^1, H_S^2, \ldots, H_S^n$ by using the possibility degree method for ranking HFLTSs. Then, we can obtain the ranking result of the alternatives.

In Step 1, we adopt Yager's linguistic quantifier method in [34] and [37] to generate the associated weights w_i of the HLOWA operator. The weights are given by the following expressions: $w_i = Q(\frac{i}{n}) - Q(\frac{i-1}{n})$, for $i = 1, 2, \ldots, n$, where

TABLE I
ASSESSMENTS PROVIDED FOR THE DECISION PROBLEM

	c_1	c_2	c_3
x_1	between vl and m	between h and vh	h
x_2	between l and m	m	lower than l
x_3	greater than h	between vl and l	greater than h

TABLE II
ASSESSMENTS TRANSFORMED INTO HFLTSS

	c_1	c_2	c_3
x_1	$\{s_1, s_2, s_3\}$	$\{s_4, s_5\}$	$\{s_4\}$
x_2	$\{s_2, s_3\}$	$\{s_3\}$	$\{s_0, s_1, s_2\}$
x_3	$\{s_4, s_5, s_6\}$	$\{s_1, s_2\}$	$\{s_4, s_5, s_6\}$

Q is a nondecreasing relative quantifier, whose membership function can be represented as

$$Q(r) = \begin{cases} 0, & r < a \\ \dfrac{r-a}{b-a}, & a \le r \le b \\ 1, & r > b \end{cases}$$

with $r \in [0, 1]$, and the parameter pair (a, b) is given. For example, the parameters (a, b) that are associated with linguistic quantifiers "most," "at least half," and "as many as possible" are $(0.3, 0.8)$, $(0, 0.5)$, and $(0.5, 1)$, respectively. The linguistic quantifier Q indicates the proportion of criteria that a decision maker requires to be satisfied by an alternative. We can generate the associated weights of the HLOWA operator to derive the overall aggregation values of alternatives according to the decision maker's requirements for criteria.

Now, we adopt the example in [20] to illustrate the aforementioned decision-making approach.

Example 4: Let $X = \{x_1, x_2, x_3\}$ be a set of alternatives; $C = \{c_1, c_2, c_3\}$ a set of criteria and $S = \{s_0$: nothing(n), s_1: very low(vl), s_2: low(l), s_3: medium(m), s_4: high(h), s_5: very high(vh), s_6: perfect(p)$\}$ a linguistic term set used to generate the linguistic expressions. The assessments given by experts to the alternatives are shown in Table I.

By the transformation function E_{G_H} defined in [20], we transform the linguistic expressions that have been provided by experts into HFLTSs which are shown in Table II.

If the decision-maker is optimistic and desires that there exists one criterion satisfied by an alternative, then the associated weighting vector w is $(1, 0, 0)^T$. Aggregating the assessments represented by HFLTSs of the alternatives x_i $(i = 1, 2, 3)$ by the HLOWA operator with $w = (1, 0, 0)^T$, we get the overall assessments $H_S^i (i = 1, 2, 3)$

$$H_S^1 = \text{HLOWA}(\{s_1, s_2, s_3\}, \{s_4, s_5\}, \{s_4\}) = \{s_4, s_5\}$$
$$H_S^2 = \text{HLOWA}(\{s_2, s_3\}, \{s_3\}, \{s_0, s_1, s_2\}) = \{s_3\}$$
$$H_S^3 = \text{HLOWA}(\{s_4, s_5, s_6\}, \{s_1, s_2\}, \{s_4, s_5, s_6\}) = \{s_4, s_5, s_6\}.$$

By the possibility degree method for ranking HFLTSs in the Appendix, we get $H_S^3 \succ^{0.6667} H_S^1 \succ^{1.0000} H_S^2$. Thus, the ranking result of the alternatives is $x_3 \succ^{0.6667} x_1 \succ^{1.0000} x_2$.

If the decision-maker is pessimistic and desires all the criteria be satisfied by an alternative, then the associated weighting vector w is $(0, 0, 1)^T$. The overall assessments $H_S^i (i = 1, 2, 3)$ of the three alternatives are

$$H_S^1 = \text{HLOWA}(\{s_1, s_2, s_3\}, \{s_4, s_5\}, \{s_4\}) = \{s_1, s_2, s_3\}$$
$$H_S^2 = \text{HLOWA}(\{s_2, s_3\}, \{s_3\}, \{s_0, s_1, s_2\}) = \{s_0, s_1, s_2\}$$
$$H_S^3 = \text{HLOWA}(\{s_4, s_5, s_6\}, \{s_1, s_2\}, \{s_4, s_5, s_6\}) = \{s_1, s_2\}.$$

Using the possibility degree method for ranking HFLTSs, we obtain the ranking of the alternatives is $x_1 \succ^{0.6667} x_3 \succ^{0.6667} x_2$, which is the same as that obtained by Rodríguez's method in [20].

If the decision-maker is neutral, then, with the HLOWA operator and its associated weighting vector $w = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})^T$, we can obtain the overall assessments $H_S^i (i = 1, 2, 3)$ of the three alternatives

$$H_S^1 = \{s_3, s_4\}, H_S^2 = \{s_2, s_3\}, H_S^3 = \{s_3, s_4, s_5\}.$$

Therefore, the ranking of the alternatives is $x_3 \succ^{0.667} x_1 \succ^{0.833} x_2$.

Comparing the aforementioned ranking results, we find that the ranking orders of alternatives are a little different. We may understand that the ranking results vary with different requirements of decision makers for criteria.

C. Multicriteria Group Decision-Making Method With Hesitant Fuzzy Linguistic Information

In this section, we will apply the HLWA and HLOWA operators to deal with the following multicriteria group decision-making problems.

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a set of alternatives, $C = \{c_1, c_2, \ldots, c_m\}$ a set of criteria, and $S = \{s_0, s_1, \ldots, s_g\}$ a linguistic term set. We let $D = \{d_1, d_2, \ldots, d_t\}$ be the set of decision makers and $R_k = (H_S^{ij^{(k)}})_{n \times m}$ be a hesitant fuzzy linguistic decision matrix, where each $H_S^{ij^{(k)}}$ is an HFLTS on S and represents the linguistic assessment provided by the decision maker $d_k \in D$ for the alternative $x_i \in X$ with respect to the criterion $u_j \in U$. The decision-makers' goal is to obtain the ranking order of the alternatives.

As was mentioned in [10], there are two basic approaches considered to obtain the overall aggregation values of alternatives. One is a direct approach

$$\{R_1, R_2, \ldots, R_t\} \to \text{solution}.$$

According to the method, a solution is derived on the basis of the individual decision matrices. The other is an indirect approach

$$\{R_1, R_2, \ldots, R_t\} \to R \to \text{solution}$$

providing the solution on the basis of an overall decision matrix. In what follows, we are going to consider a direct method. Based on the HLWA operator and the HLOWA operator, we give aggregation ways associated with different decision information of criteria or experts. Then, we apply the comparison method in the Appendix for ranking the overall aggregation values of alternatives. The specific method is as follows.

Step 1: According to the different situations where importance weights of criteria are known or unknown, we utilize the HLWA operator or the HLOWA operator to derive the individual overall aggregation values $H_i^{(k)}$ $(i = 1, 2, \ldots, n;\ k = 1, 2, \ldots, t)$ of alternatives x_i $(i = 1, 2, \ldots, n)$, that is

$$H_i^{(k)} = \text{HLOWA}\left(H_S^{i1\,(k)}, H_S^{i2\,(k)}, \ldots, H_S^{im\,(k)}\right)$$

or

$$H_i^{(k)} = \text{HLWA}\left(H_S^{i1\,(k)}, H_S^{i2\,(k)}, \ldots, H_S^{im\,(k)}\right).$$

Step 2: If the importance weights of experts are unknown, then we utilize the HLOWA operator to derive the overall aggregation values H_i $(i = 1, 2, \ldots, n)$ of alternatives x_i $(i = 1, 2, \ldots, n)$, where

$$H_i = \text{HLOWA}\left(H_i^{(1)}, H_i^{(2)}, \ldots, H_i^{(t)}\right).$$

If each expert plays a different role and we know the relative importance weighting vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)^T$ of experts such that $\lambda_j \geq 0$ and $\sum_{j=1}^{t} \lambda_j = 1$, then we utilize the HLWA operator to derive the overall aggregation values H_i $(i = 1, 2, \ldots, n)$ of alternatives x_i $(i = 1, 2, \ldots, n)$

$$H_i = \text{HLWA}\left(H_i^{(1)}, H_i^{(2)}, \ldots, H_i^{(t)}\right).$$

Step 3: Compare $H_i (i = 1, 2, \ldots, n)$ by using the possibility degree method in the Appendix, rank the alternatives x_i $(i = 1, 2, \ldots, n)$, then select the best one(s).

Example 5: A practical application of the proposed approaches involves the evaluation of university faculty for tenure and promotion. The criteria used at some universities are teaching (u_1), research (u_2), and service (u_3) whose weighting vector is $w = (0.4, 0.3, 0.3)^T$. Suppose there are five candidates x_i $(i = 1, 2, 3, 4, 5)$ to be evaluated by three experts $d_k (k = 1, 2, 3)$ under these three attributes. We suppose the label set $S = \{s_0\text{: nothing}, s_1\text{: very low}, s_2\text{: low}, s_3\text{: medium}, s_4\text{: high}, s_5\text{: very high}, s_6\text{: perfect}\}$ and assume that the decision-making matrices $R_k = (r_{ij}^{(k)})_{5\times 3}$ $(k = 1, 2, 3)$ are as follows:

$$R_1 = \begin{pmatrix} \{s_4, s_5\} & \{s_3\} & \{s_4\} \\ \{s_2\} & \{s_5\} & \{s_3\} \\ \{s_1\} & \{s_3, s_4\} & \{s_1, s_2\} \\ \{s_5, s_6\} & \{s_4\} & \{s_3\} \\ \{s_1\} & \{s_1, s_2\} & \{s_5\} \end{pmatrix}$$

$$R_2 = \begin{pmatrix} \{s_5, s_6\} & \{s_2\} & \{s_3, s_4\} \\ \{s_3, s_4\} & \{s_4, s_5\} & \{s_2\} \\ \{s_2\} & \{s_2, s_3\} & \{s_1\} \\ \{s_5, s_6\} & \{s_4, s_5, s_6\} & \{s_3, s_4, s_5\} \\ \{s_2\} & \{s_1\} & \{s_4\} \end{pmatrix}$$

$$R_3 = \begin{pmatrix} \{s_5\} & \{s_4, s_5\} & \{s_6\} \\ \{s_4\} & \{s_3, s_4\} & \{s_3\} \\ \{s_3\} & \{s_1, s_2\} & \{s_2\} \\ \{s_5\} & \{s_6\} & \{s_4\} \\ \{s_1, s_2\} & \{s_3\} & \{s_4\} \end{pmatrix}.$$

Step 1: Since the weights of criteria are given, we utilize the HLWA operator to aggregate the decision matrices R_k $(k = 1, 2, 3)$ to derive the individual overall aggregation values $H_i^{(k)}$ $(i = 1, 2, 3, 4, 5)$ of the alternatives $x_i (i = 1, 2, 3, 4, 5)$

$$H_1^{(1)} = \{s_4\}, \quad H_2^{(1)} = \{s_3\}, \quad H_3^{(1)} = \{s_2\}$$
$$H_4^{(1)} = \{s_4, s_5\}, \quad H_5^{(1)} = \{s_2, s_3\}, \quad H_1^{(2)} = \{s_4\}$$
$$H_2^{(2)} = \{s_3, s_4\}, \quad H_3^{(2)} = \{s_2\}, \quad H_4^{(2)} = \{s_4, s_5, s_6\}$$
$$H_5^{(2)} = \{s_3\}, \quad H_1^{(3)} = \{s_5, s_6\}, \quad H_2^{(3)} = \{s_3, s_4\}$$
$$H_3^{(3)} = \{s_2\}, \quad H_4^{(3)} = \{s_5\}, \quad H_5^{(3)} = \{s_3\}.$$

Step 2: Utilize the HLOWA operator to derive the overall aggregation values of the alternatives x_i $(i = 1, 2, 3, 4, 5)$. We shall assume the quantifier guiding this aggregation to be "as many as possible" with the pair (0.5,1). Its associated fuzzy set is

$$Q(r) = \begin{cases} 0, & 0 \leq r < 0.5, \\ 2r - 1, & 0.5 \leq r \leq 1. \end{cases}$$

Therefore, we can compute the associated HLOWA weights w_i $(i = 1, 2, 3)$: $w_1 = Q(\frac{1}{3}) - Q(0) = 0, w_2 = Q(\frac{2}{3}) - Q(\frac{2}{3}) = \frac{1}{3}$, and $w_3 = 1 - Q(\frac{2}{3}) = \frac{2}{3}$. With $w = (0, \frac{1}{3}, \frac{2}{3})^T$ and the HLOWA operator, we get the overall aggregation values H_i of the alternatives x_i for $i = 1, 2, 3, 4, 5$

$$H_1 = \text{HLOWA}(H_1^{(1)}, H_1^{(2)}, H_1^{(3)}) = \{s_4\}$$
$$H_2 = \text{HLOWA}(H_2^{(1)}, H_2^{(2)}, H_2^{(3)}) = \{s_3\}$$
$$H_3 = \text{HLOWA}(H_3^{(1)}, H_3^{(2)}, H_3^{(3)}) = \{s_2\}$$
$$H_4 = \text{HLOWA}(H_4^{(1)}, H_4^{(2)}, H_4^{(3)}) = \{s_4, s_5\}$$
$$H_5 = \text{HLOWA}(H_5^{(1)}, H_5^{(2)}, H_5^{(3)}) = \{s_2, s_3\}.$$

Step 3: Using the possibility degree method in the Appendix, we compare the HFLTSs H_i $(i = 1, 2, 3, 4, 5)$. By Step 1 and Step 2 in the Appendix, we construct the possibility degree matrix P and the preference relation matrix U

$$P = \begin{pmatrix} \frac{1}{2} & 1 & 1 & \frac{1}{4} & 1 \\ 0 & \frac{1}{2} & 1 & 0 & \frac{3}{4} \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{4} \\ \frac{3}{4} & 1 & 1 & \frac{1}{2} & 1 \\ 0 & \frac{1}{4} & \frac{3}{4} & 0 & \frac{1}{2} \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

By Step 3 in the Appendix, we get $V_1 = \{H_4,\}, V_2 = \{H_1\}$, $V_3 = \{H_2\}, V_4 = \{H_5\}$, and $V_5 = \{H_3\}$. Thus, the ranking result of the HFLTSs is $H_4 \succ^{\frac{3}{4}} H_1 \succ^1 H_2 \succ^{\frac{3}{4}} H_5 \succ^{\frac{3}{4}} H_3$. Hence, we obtain the ranking of the alternatives x_i $(i = 1,2,3,4,5)$: $x_4 \succ^{\frac{3}{4}} x_1 \succ^1 x_2 \succ^{\frac{3}{4}} x_5 \succ^{\frac{3}{4}} x_3$.

VI. Conclusion

The theory of HFLTSs is very useful in objectively dealing with the situations in which there is hesitancy in providing linguistic assessments. The existing comparison methods and aggregation theory are limited in their applications; hence, the importance of studying more suitable ones, which is the focus of this paper.

Thus, two new comparison methods have been proposed for HFLTSs. Compared with the method in [20], which uses the comparison theory of numerical intervals to compare HFLTSs, our methods are based on the probability theory and sufficiently consider the property that an HFLTS consists of finite linguistic terms. Therefore, our comparison results are more reasonable especially for the case in which two HFLTSs have one common element. We have developed two aggregation operators, an HLWA operator and an HLOWA operator, by defining a convex combination operation on HFLTSs. Based on the two aggregation operators and the comparison theory for HFLTSs, decision-making methods have been proposed to deal with MCDM problems in which the assessments of alternatives under criteria are represented by HFLTSs. These methods can be used to deal with different decision-making situations, where the weights of criteria or experts can be known or unknown. Moreover, by using these methods, we can choose suitable HFOWA weighting vectors to reflect different attitudes of a decision maker or his/her requirements for criteria or for experts.

The aggregation method in this paper can be used to aggregate HFLTSs and their associated numerical weights. In future work, we will study the HFLTS information aggregations in more general contexts, such as the situation with the linguistic weights of the arguments. Following our previous work in [26], we will also consider how to assess criteria or expert weights according to the assessments, as represented by HFLTSs, and develop more decision-making methods for MCDM problems with HFLTSs information.

Appendix

In this Appendix, we use the possibility degree formulas (1) and (2) to introduce a possibility degree method for ranking n HFLTSs in a similar way to the method in [27]. Let S be a linguistic term set and $H_S^1, H_S^2, \ldots, H_S^n$ be n HFLTSs on S. By the following steps, we can rank these HFLTSs.

Step 1: By pairwise comparisons among these n HFLTSs, we construct a possibility degree matrix

$$P = \begin{pmatrix} 0.5 & p_{12} & . & . & . & p_{1n} \\ p_{21} & 0.5 & . & . & . & p_{2n} \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ p_{n1} & p_{n2} & . & . & . & 0.5 \end{pmatrix}$$

where $p_{ij} = p(H_S^i \geq H_S^j)$ is calculated by (1) or (2).

Step 2: Construct the preference relation matrix $U = (u_{ij})$ from the possibility degree matrix P, where, for any i, j

$$u_{ij} = \begin{cases} 1, & p_{ij} \geq 0.5, \\ 0, & p_{ij} < 0.5. \end{cases}$$

Step 3: Find all the rows in which the elements are all equal to 1 in U. We label these rows $V = \{j_1, j_2, \ldots, j_t\}$. From the complementarity of the possibility degree formula, we can easily obtain that the corresponding compared HFLTSs $H_S^{j_1}, H_S^{j_2}, \ldots, H_S^{j_t}$ are indifferent. Let $V_1 = \{H_S^{j_1}, H_S^{j_2}, \ldots, H_S^{j_t}\}$. Remove the elements in rows $j_1, j_2, \ldots, j_t$ and columns $j_1, j_2, \ldots, j_t$ from the matrix U, and the remained elements construct a matrix U_1. Then, find the rows in which the elements are all equal to 1 in U_1, and denote by V_2 the set of corresponding HFLTSs, which are also indifferent. Repeating the process, we can divide the set of n HFLTSs into $V_1, V_2, \ldots, V_l$.

Step 4: If each V_i has only one element $H_S^{k_i}$, then the rank of $H_S^1, H_S^2, \ldots, H_S^n$ is

$$H_S^{k_1} \succ^{p(H_S^{k_1} > H_S^{k_2})} H_S^{k_2} \succ^{p(H_S^{k_2} > H_S^{k_3})} \cdots \succ^{p(H_S^{k_{n-1}} > H_S^{k_n})} H_S^{k_n}.$$

Suppose there is some V_i containing more than one HFLTS. Then, these HFLTSs are indifferent, that is, for any two HFLTSs $H_S^{i_1}$ and $H_S^{i_2}$ in V_i, we have $H_S^{i_1} \sim H_S^{i_2}$. We can further compare these HFLTSs in V_i as follows:

If $\mathrm{Ind}({H_S^{i_1}}^+) - \mathrm{Ind}({H_S^{i_1}}^-) > \mathrm{Ind}({H_S^{i_2}}^+) - \mathrm{Ind}({H_S^{i_2}}^-)$, then $H_S^{i_2}$ is said to be quasi-superior to $H_S^{i_1}$, denoted by $H_S^{i_1} \rhd H_S^{i_2}$. If $\mathrm{Ind}({H_S^{i_1}}^+) - \mathrm{Ind}({H_S^{i_1}}^-) = \mathrm{Ind}({H_S^{i_2}}^+) - \mathrm{Ind}({H_S^{i_2}}^-)$, then we have $H_S^{i_1} = H_S^{i_2}$.

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