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Entropy measures for hesitant fuzzy sets and their application in multi-criteria decision-making

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Abstract. Entropy is used to measure the uncertain degree of fuzzy sets and has been widely used in many fields. This paper introduces an axiomatic definition of entropy measure and a novel entropy formula for hesitant fuzzy elements (HFEs). Afterwards, a general form of entropy measures for HFEs is proposed, from which a family of concrete entropy formulas for HFEs can be derived. Compared with the existing ones, these formulas can measure both fuzziness and hesitation of HFEs, as a result, the uncertain information can be described in a more appropriate manner. The proposed axiomatic definition and entropy formulas are used to define the entropy measure for hesitant fuzzy sets. A multi-criteria decision making model which uses the proposed entropy measures for HFEs to compute the criteria weights and obtain a ranking of alternatives is introduced.

Keywords: Hesitant fuzzy set, entropy measure, multi-criteria decision-making

1. Introduction

Since Zadeh [35] introduced fuzzy sets, many generalized forms have been proposed, among which there are interval-valued fuzzy sets (IVFSs) [3], intuitionistic fuzzy sets (IFSs) [1], interval-value intuitionistic fuzzy sets (IVIFS) [2], vague sets [10], type-2 fuzzy sets [17], type-n fuzzy sets [7] and fuzzy multisets [18]. All these extensions are based on the same rationale that defining a fuzzy set is not an easy task because there is no a clear way to assign the membership degree of an element to a fixed set [23, 24].

Recently, a new extension of fuzzy sets, so-called Hesitant Fuzzy Set (HFS), has been proposed in [23, 24]. The motivation is that when defining the membership of an element, the difficulty of establishing the membership degree is not because there is a margin of error, or some possibility distribution on the possible values, but because there is a set of possible values [23]. HFSs can be used when experts hesitate among several values to provide their preferences over the alternatives and the use of only one value is not enough to reflect their knowledge in a suitable way.

In the last years, HFSs have attracted much attention of researchers and different proposals have been introduced in the literature such as, distance measures, similarity measures, and entropy measures [14, 31, 32], different aggregation operators [12, 13, 29], decision making methods that deal with HFS [4, 26–28], even it has been extended to deal with linguistic information [15, 16, 20].

This paper is focused on the entropy measure for HFSs that is a very important notion for measuring uncertain information, along with a lot of

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research productions of entropy measures for FSs and IFSs. Zadeh first used fuzziness in an entropy measure which was mentioned in [35]. Then, De Luca and Termini [6] proposed the axioms that the fuzzy entropy should comply and they defined the entropy of a fuzzy set based on Shannon's function. Yager [33] presented an entropy measure to view the fuzziness degree of the fuzzy set in terms of a lack of distinction between the fuzzy set and its complement. Other entropies for fuzzy sets with different points of views can be found in [8, 19, 22].

In hesitant fuzzy environment, Xu and Xia [32] proposed an entropy axiomatic definition and some formulas for hesitant fuzzy elements (HFEs). Farhadinia [9] presented some counterexamples to point out that the entropies proposed in [32] cannot discriminate some HFEs, even though they are apparently different and proposed a series of entropy measures for HFSs, which are based on the distance measures for HFSs. However, it can be shown that Farhadinia's entropy measures fail to discriminate HFEs that have the same distance with the HFE $\{0.5\}$. Therefore, in order to overcome this shortcoming, this paper aims to propose new entropy axiomatic definitions and construct entropy formulas of HFEs and HFSs, which can reflect the uncertainty of HFEs and HFSs in a proper way. Based on these entropy formulas, an approach to solve multi-criteria decision-making problems with unknown criteria weights is introduced.

HFS is a simple and effective tool used to express experts' hesitation in decision-making. Suppose two experts discuss the membership degree of an alternative for a criterion, and one of them wants to assign 0.1 and the other 0.8. Then the membership degree of the alternative for the criterion might be described by the HFE $\{0.1, 0.8\}$, which describes the uncertain information of experts' preference. Therefore, it is necessary to depict the uncertainty degree associated to the HFE $\{0.1, 0.8\}$ in an appropriate way. Since HFS is a generalized form of FS, the fuzziness as a characteristic of FS, is also an important index for HFS. Besides, HFE has its own characteristic, that is the hesitation among the possible membership degrees 0.1 and 0.8. Thus, in order to depict the uncertainty reflected by an HFE, the fuzziness and hesitation of information should be considered. Therefore, the proposed axiomatic definitions and entropy measures can depict both fuzziness and hesitation of HFEs and HFSs.

The rest of this paper is organized as follows. Section 2 reviews some concepts of HFSs and distance measures of HFSs. Section 3 proposes a new axiomatic definition and an entropy measure for HFEs. Subsequently, a family of entropy measures based on the previous one is introduced. These proposals are generalized to define an axiomatic definition and entropy measures for HFSs. Section 4 compares the proposed axiomatic definition of entropy and concrete entropy formulas for HFEs with those defined in [9, 32]. Section 5 puts forward a method for multi-criteria decision-making which uses entropy measures for HFSs to obtain a ranking of alternatives. Section 6 introduces an illustrative example to demonstrate the practicality and effectiveness of the proposed formulas in a multi-criteria decision-making problem. Finally, this paper is concluded in Section 7.

2. Preliminaries

This section reviews some basic concepts, such as Hesitant Fuzzy Sets (HFSs) [23] and distance measures for HFSs [31]. Throughout this paper, $X = \{x_1, x_2, \ldots, x_n\}$ is used to denote the universe of discourse.

In [23] it was introduced a new extension of fuzzy set with the goal of modelling the uncertainty originated by the hesitation that might arise in the assignment of the membership degrees of the elements to a fuzzy set.

Definition 1. [23] Let X be a discourse set. A HFS on X is defined in terms of a function h that when is applied to X, it returns a non-empty subset of values in the interval $[0, 1]$.

To be easily understood, Xia and Xu [30] expressed an HFS by

$$A = \{\langle x_i, h_A(x_i)\rangle | x_i \in X\}, \tag{1}$$

where $h_A(x_i)$ is a set of some values in $[0, 1]$, denoting the possible membership degrees of the element $x_i \in X$ to the set A. Especially, if there is only one value in each $h_A(x_i)(i = 1, 2, \ldots, n)$, then the HFS A is reduced to a fuzzy set, which indicates that fuzzy sets are a special type of HFSs. HFS(X) is the set of all the HFSs on X.

For convenience, Xia and Xu [30] named $h_A(x_i)$, abbreviated to h, a hesitant fuzzy element (HFE). For any HFE $h = \{h^1, h^2, \ldots, h^{l_h}\}$, we assume that $h^i < h^j$, for $\forall i < j (i, j = 1, 2, \ldots, l_h)$ in the whole paper, where l_h, abbreviated to l in the case of no confusion, is the number of values in h and is called the length of h. H is the set of all the HFEs

Example 1. Let $X = \{x_1, x_2, x_3\}$ be the discourse set. Then $A = \{\langle x_1, \{0.2, 0.3\}\rangle, \langle x_2, \{0.1, 0.2\}\rangle, \langle x_3, \{0.6, 0.7, 0.8\}\rangle\}$ is an HFS on X.

Let $A = \{\langle x_j, h_A(x_j)\rangle | x_j \in X\}$ and $B = \{\langle x_j, h_B(x_j)\rangle | x_j \in X\}$ be two HFSs. In general, the lengths of $h_A(x_j)$ and $h_B(x_j)$ may be different. Let $l_{x_j} = max\{l_{h_A(x_j)}, l_{h_B(x_j)}\}$. In order to operate correctly, $h_A(x_j)$ and $h_B(x_j)$ should have the same length l_{x_j}. To do this, the shorter one is extended until the lengths of both are the same. The best way to extend the shorter one is to add the same element in it until the changed HFE has the same length as the longer one. Any value in the shorter one might be added to extend it. In this paper, it is considered an optimistic point of view in which the shorter one is extended by repeating its maximum element.

For two HFSs A and B, we assume that $h_A(x_j)$ and $h_B(x_j)$ have the same length l_{x_j}. Let $h^i_A(x_j)$ and $h^i_B(x_j)$ be the ith smallest values in $h_A(x_j)$ and $h_B(x_j)$, respectively. Xu and Xia [31] gave a variety of distance measures for HFSs, some of which are described as follows:

– The generalized hesitant normalized distance:

$$d_{gh}(A, B) = \left\{ \frac{1}{n} \sum_{j=1}^{n} \left(\frac{1}{l_{x_j}} \sum_{i=1}^{l_{x_j}} \left| h^i_A(x_j) - h^i_B(x_j) \right|^{\lambda} \right) \right\}^{\frac{1}{\lambda}} \quad (2)$$

where $\lambda > 0$.

– The generalized hesitant weighted distance:

$$d_{ghw}(A, B) = \left\{ \sum_{j=1}^{n} \omega_j \left(\frac{1}{l_{x_j}} \sum_{i=1}^{l_{x_j}} \left| h^i_A(x_j) - h^i_B(x_j) \right|^{\lambda} \right) \right\}^{\frac{1}{\lambda}} \quad (3)$$

where $\lambda > 0$, and $\omega_j (j = 1, 2, \ldots, n)$ is the weight of the element x_j with $\omega_j \in [0, 1]$ and $\sum_{j=1}^{n} \omega_j = 1$.

More information about distance measures for HFS can be found in [21, 31].

3. Entropy measure for HFSs

In order to investigate new entropy measures for HFSs, first some entropy measures for HFEs are studied and a new axiomatic definition of the entropy measure for HFEs is proposed. Furthermore, we put forward a family of entropy measures and a series of entropy formulas for HFEs. Finally, entropy measures for HFSs based on the entropy measures for HFEs are proposed. The efficiency of the proposed entropy measures is demonstrated through comparisons with some existing entropy measures in [9] and [31].

3.1. Entropy measures for HFEs

For an HFE $h = \{h^1, h^2, \ldots, h^l\}$, its uncertainty of information should include two facts: the fuzziness and hesitation of information. The fuzziness is dominated by the difference between the averaging value of the elements in h and the most fuzzy value $\{0.5\}$. The hesitation is reflected by the deviation degree of the elements in h. The averaging value and the deviation function value of h are defined as follows.

Definition 2. [12] Let $h = \{h^1, h^2, \ldots, h^l\}$ be an HFE, then

1. the averaging value (score function value) of an HFE h is defined as

$$\theta(h) = \frac{1}{l} \sum_{i=1}^{l} h^i. \quad (4)$$

2. the deviation function value of an HFE h is defined as

$$\eta(h) = \frac{2}{l(l-1)} \sum_{i=1}^{l-1} \sum_{j=i+1}^{l} (h^j - h^i). \quad (5)$$

Based on the score function $\theta(h)$ and the deviation function $\eta(h)$, an entropy axiomatic definition for HFEs is proposed.

Definition 3. The entropy on an HFE h is a real-valued function $E : H \to [0, 1]$, satisfying the following axiomatic requirements:

(E1) $E(h) = 0$, if and only if $h = \{0\}$ or $h = \{1\}$;
(E2) $E(h) = 1$, if and only if $\theta(h) = 0.5$;
(E3) $E(h_1) \leq E(h_2)$, if $\theta(h_1) \leq \theta(h_2)$ and $\eta(h_1) \leq \eta(h_2)$ for $\theta(h_2) \leq 0.5$, or $\theta(h_1) \geq \theta(h_2)$ and $\eta(h_1) \leq \eta(h_2)$ for $\theta(h_2) \geq 0.5$;
(E4) $E(h) = E(h^c)$, where h^c is the complement of an HFE h and is defined by $h^c = \bigcup_{\gamma \in h} \{1 - \gamma\}$.

Remark 1. The axiomatic requirement (E3) reflects that for any two HFEs h_1 and h_2, if $\theta(h_1)$ is further from 0.5 than $\theta(h_2)$, that is, $|\theta(h_1) - 0.5| > |\theta(h_2) - 0.5|$, and the deviation function value of h_1 is less than $\eta(h_2)$, then the uncertainty of h_1 should be less than h_2.

The new entropy measure for HFEs is defined and its properties are studied.

Theorem 1. For each HFE $h \in H$, $E(h)$ is defined by

$$E(h) = \frac{1 - |\cos(\theta(h) \cdot \pi)| + \eta(h)}{1 + \eta(h)}. \tag{6}$$

Then E is an entropy measure for HFEs.

Proof. It is sufficient to show the mapping $E(h)$, defined by Equation 6, for satisfying the conditions (E1)–(E4) in Definition 3.

Since $\theta(h) = \frac{1}{l}\sum_{i=1}^{l} h^i$ and $\eta(h) = \frac{2}{l(l-1)}\sum_{i=1}^{l-1}\sum_{j=i+1}^{l}(h^j - h^i)$ for an HFE $h = \{h^1, h^2, \ldots, h^l\}$, we have $0 \le \theta(h) \le 1$, $0 \le \eta(h) \le 1$. Hence,

$$0 \le \frac{1 - |\cos(\theta(h) \cdot \pi)| + \eta(h)}{1 + \eta(h)} \le 1.$$

(E1) $E(h) = \frac{1-|\cos(\theta(h)\cdot\pi)|+\eta(h)}{1+\eta(h)} = 0$ if and only if $|\cos(\theta(h) \cdot \pi)| - \eta(h) = 1$. Since $0 \le \eta(h) \le 1$ and $0 \le |\cos(\theta(h) \cdot \pi)| \le 1$, we have $-1 \le |\cos(\theta(h) \cdot \pi)| - \eta(h) \le 1$. Therefore, $|\cos(\theta(h) \cdot \pi)| - \eta(h) = 1$ if and only if $\eta(h) = 0$ and $|\cos(\theta(h) \cdot \pi)| = 1$, i.e., $h = \{0\}$ or $h = \{1\}$.

(E2) $E(h) = \frac{1-|\cos(\theta(h)\cdot\pi)|+\eta(h)}{1+\eta(h)} = 1$ if and only if $\cos(\theta(h) \cdot \pi) = 0$, that is, $\theta(h) = 0.5$.

(E3) Let $\theta(h) = \theta$, $\eta(h) = \eta$, the Equation 6 can be noted as $E(h) = E(\theta, \eta) = \frac{1-|\cos(\theta\pi)|+\eta}{1+\eta}$.
The partial derivative of $E(\theta, \eta)$ with respect to θ is as follows,

$$\frac{\partial E(\theta, \eta)}{\partial \theta} = \begin{cases} \frac{\pi\sin(\theta\pi)(1+\eta)}{(1+\eta)^2}, & 0 < \theta \le 0.5, \\ \frac{-\pi\sin(\theta\pi)(1+\eta)}{(1+\eta)^2}, & 0.5 \le \theta < 1. \end{cases}$$

Since $0 < \sin(\theta\pi) \le 1$ and $1 \le 1 + \eta \le 2$ for $0 < \theta < 1$, we can get $\frac{\partial E(\theta,\eta)}{\partial\theta} > 0$ if $0 < \theta \le 0.5$; $\frac{\partial E(\theta,\eta)}{\partial\theta} < 0$ if $0.5 \le \theta < 1$. Therefore, $E(\theta, \eta)$ is strictly monotone increasing with respect to $\theta \in (0, 0.5]$, and strictly monotone decreasing with respect to $\theta \in [0.5, 1)$.
On the other hand, the partial derivative of $E(\theta, \eta)$ with respect to $\eta \in [0, 1]$ is denoted as follows,

$$\frac{\partial E(\theta, \eta)}{\partial \eta} = \frac{|\cos(\theta\pi)|}{(1 + \eta)^2}.$$

Clearly, $\frac{\partial E(\theta,\eta)}{\partial\eta} \ge 0$ and $E(\theta, \eta)$ is monotonically increasing with respect to $\eta \in [0, 1]$.
From the above discussion, it is easy to get that if $\theta(h_1) \le \theta(h_2) \le 0.5$ and $\eta(h_1) \le \eta(h_2)$, then $E(\theta(h_1), \eta(h_1)) \le E(\theta(h_1), \eta(h_2)) \le E(\theta(h_2), \eta(h_2))$, that is, $E(h_1) \le E(h_2)$; if $\theta(h_1) \ge \theta(h_2) \ge 0.5$ and $\eta(h_1) \le \eta(h_2)$, then $E(\theta(h_1), \eta(h_1)) \le E(\theta(h_1), eta(h_2)) \le E(\theta(h_2), \eta(h_2))$, that is, $E(h_1) \le E(h_2)$.

(E4) For an HFE $h = \{h^1, h^2, \ldots, h^l\}$, we have $\theta(h^c) = \frac{1}{l}\sum_{i=1}^{l}(1 - h^i) = 1 - \theta(h)$ and $\eta(h^c) = (1 - h^l) - (1 - h^1) = \eta(h)$. Therefore,

$$\begin{aligned} E(h^c) &= \frac{1 - |\cos(\theta(h^c) \cdot \pi)| + \eta(h^c)}{1 + \eta(h^c)} \\ &= \frac{1 - |\cos((1 - \theta(h)) \cdot \pi)| + \eta(h)}{1 + \eta(h)} \\ &= \frac{1 - |-\cos(\theta(h) \cdot \pi)| + \eta(h)}{1 + \eta(h)} \\ &= \frac{1 - |\cos(\theta(h) \cdot \pi)| + \eta(h)}{1 + \eta(h)} = E(h). \end{aligned}$$

Equation 6 satisfies the conditions (E1)–(E4) in Definition 4, thus E is an entropy measure for HFEs.

Corollary 1. *$E(h)$ is strictly monotone increasing with respect to $\theta \in (0, 0.5]$ and strictly monotone decreasing with respect to $\theta \in [0.5, 1)$. In addition, it is monotonically increasing with respect to $\eta \in [0, 1]$.*

The conclusion is obviously shown in the proof process of Theorem 1.

From Corollary 1, we know that the greater the difference between $\theta(h)$ and 0.5 is, the greater $E(h)$ is; the greater the deviation function value $\eta(h)$ is, the greater $E(h)$ is. Since the fuzziness of an HFE h is dominated by the difference between the score function value $\theta(h)$ and the most fuzzy value 0.5, and the hesitation is reflected by the deviation function value $\eta(h)$, it is obvious that the entropy measure $E(h)$ considers both the fuzziness and hesitation of an HFE h.

3.2. A family of entropy measures for HFEs

In this subsection, a family of entropy measures for HFEs based on the entropy measure E is presented. Additionally, four concrete entropy measures are studied.

Theorem 2. For each HFE h, let

$$E_g(h) = \frac{f(\theta(h)) + k\eta(h)}{1 + k\eta(h)}, \quad (7)$$

where $k \in [0, 1]$. Then $E_g(h)$ is an entropy measure for an HFE h, where the function $f : [0, 1] \to [0, 1]$ satisfies the following three conditions:

1. $f(1 - x) = f(x)$;
2. $f(x)$ is strictly monotone increasing with respect to $x \in (0, 0.5]$ and strictly monotone decreasing with respect to $x \in [0.5, 1)$;
3. It interpolates three points, $(0, 0)$, $(\frac{1}{2}, 1)$ and $(1, 0)$.

Corollary 2. *$E_g(h)$ is monotonically increasing with respect to $\theta \in (0, 0.5]$, and monotonically decreasing with respect to $\theta \in [0.5, 1)$. In addition, it is monotonically increasing with respect to $\eta \in [0, 1]$.*

It is noted that if we change the function $f(x)$ in $E_g(h)$ defined by Equation 7, we can obtain a series of entropy measures for HFEs. For instance, let $k = 1$ and let $f(x) = 1 - |1 - 2x|$, $f(x) = 1 - 4(x - \frac{1}{2})^2$, $f(x) = 1 - |\cos(\pi x)|$ and $f(x) = \sin(\pi x)$, respectively.

Four different entropy formulas are then obtained:

$$E_{g1}(h) = \frac{1 - |1 - 2\theta(h)| + \eta(h)}{1 + \eta(h)}, \quad (8)$$

$$E_{g2}(h) = E(h) = \frac{1 - |\cos(\theta(h) \cdot \pi)| + \eta(h)}{1 + \eta(h)}, \quad (9)$$

$$E_{g3}(h) = E_2(h) = \frac{\sin(\theta(h) \cdot \pi) + \eta(h)}{1 + \eta(h)}, \quad (10)$$

$$E_{g4}(h) = \frac{1 - 4(\theta(h) - \frac{1}{2})^2 + \eta(h)}{1 + \eta(h)}. \quad (11)$$

Figure 1 shows the graphs of the four entropy formulas, $E_{g1}(h)$, $E_{g2}(h)$, $E_{g3}(h)$ and $E_{g4}(h)$. To do so, $\eta(h)$ is a fixed value.

From Fig. 1, it is easy to see that $E_{g1}(h)$, $E_{g2}(h)$, $E_{g3}(h)$ and $E_{g4}(h)$ are all monotonically increasing with respect to $\theta \in (0, 0.5]$, and monotonically decreasing with respect to $\theta \in [0.5, 1)$. However, the increasing degree and decreasing degree of the four functions with respect to θ are different. When $\theta(h)$ closes to 0.5, the value of $E_{g3}(h)$ increases slowly

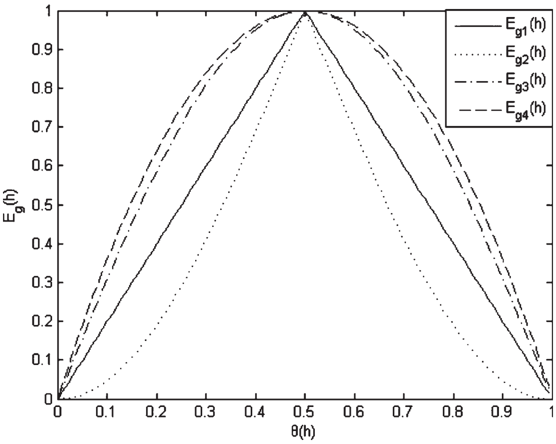

Fig. 1. Graphic representation of $E_{g1}(h)$, $E_{g2}(h)$, $E_{g3}(h)$ and $E_{g4}(h)$.

for $\theta(h) \leq 0.5$ and decreases slowly for $\theta(h) > 0.5$. This type of entropy measures such as $E_{g3}(h)$ is called conservative. The graph of $E_{g4}(h)$ is similar to $E_{g3}(h)$, so $E_{g4}(h)$ also belongs to this type. Compared with $E_{g3}(h)$, $E_{g2}(h)$ has the opposite characteristic. When $\theta(h)$ closes to 0.5, the value of $E_{g2}(h)$ increases quickly for $\theta(h) \leq 0.5$ and decreases quickly for $\theta(h) > 0.5$. Simultaneously, when $\theta(h)$ stays away from 0.5, the value of $E_{g2}(h)$ increases slowly for $\theta(h) \leq 0.5$ and decreases slowly for $\theta(h) > 0.5$. This type of entropy measures such as $E_{g2}(h)$ is called risker. Regarding $E_{g1}(h)$, its graph is a solid line which increases and decreases steadily. This type of entropy measure, $E_{g1}(h)$, is called independent. Therefore, different type of entropy measures can be chosen according to the attitude of a decision-maker for the value $\theta(h)$.

3.3. Entropy measures for HFSs

HFEs are the basic elements of HFSs. According to the entropy measures of HFEs, the entropy axiomatic definition and some concrete entropy formulas for HFSs are proposed.

Definition 4. A real-valued function E : HFS(X)$\to$ $[0, 1]$ is called an entropy measure for HFSs, if it satisfies the following axiomatic requirements.

Let $A = \{\langle x_i, h_A(x_i)\rangle | x_i \in X\}$ and $B = \{\langle x_j, h_B(x_j)\rangle | x_j \in X\}$ be two HFSs,

(E1) $E(A) = 0$ if and only if $h_A(x_i) = \{0\}$ or $h_A(x_i) = \{1\}$ for all $x_i \in X$;
(E2) $E(A) = 1$ if and only if $\theta(h_A(x_i)) = 0.5$ for all $x_i \in X$;

(E3) $E(A) \leq E(B)$, if $\theta(h_A(x_i)) \leq \theta(h_B(x_i))$ and $\eta(h_A(x_i)) \leq \eta(h_B(x_i))$ for $\theta(h_B(x_i)) \leq 0.5$, or $\theta(h_A(x_i)) \geq \theta(h_B(x_i))$ and $\eta(h_A(x_i)) \leq \eta(h_B(x_i))$ for $\theta(h_B(x_i)) \geq 0.5$ for any $x_i \in X$;
(E4) $E(A) = E(A^c)$, where A^c is the complement of an HFS A on X, defined by $A^c = \{\langle x_i, h_A^c(x_i)\rangle | x_i \in X\}$.

Theorem 3. Let $A = \{\langle x_i, h_A(x_i)\rangle | x_i \in X\}$ be an HFS on X. Then

$$\overline{E}(A) = \frac{1}{n}\sum_{i=1}^{n} E(h_A(x_i)), \tag{12}$$

is an entropy measure for the HFS A, where $E(\cdot)$ is an entropy measure for HFEs.
Proof. Since E satisfies the requirements in Definition 3, it is easy to show that $\overline{E}$ satisfies the requirements in Definition 4.

From Theorems 2 and 3, for HFS $A = \{\langle x_i, h_A(x_i)\rangle | x_i \in X\}$, a family of entropy measures is obtained:

$$\overline{E}_g(A) = \frac{1}{n}\sum_{i=1}^{n} \frac{f(\theta(h(x_i))) + k\eta(h(x_i))}{1 + k\eta(h(x_i))}, \tag{13}$$

where $k \in [0, 1]$ and the function $f(x) : [0, 1] \to [0, 1]$ satisfies $f(1 - x) = f(x)$, and is strictly monotone increasing with respect to $x \in (0, 0.5]$ and strictly monotone decreasing with respect to $x \in [0.5, 1)$. Besides, it interpolates three points, $(0, 0)$, $(\frac{1}{2}, 1)$ and $(1, 0)$. Taking some special functions $f(x)$ such as the ones used in Subsection 3.2. for HFEs, we can get different entropy measures for HFSs.

4. Comparisons with existing entropy axiomatic definitions and entropy measures for HFEs

This section introduces a comparison between the proposed axiomatic Definition 3 and the entropy measures with those presented in [9] and [32]. Several examples are shown to illustrate that the proposed entropy measures can reflect both fuzziness and hesitation of HFEs.

4.1. Theoretical analysis

Xu and Xia [32] proposed the following axiomatic definition and entropy formulas for HFEs.

Definition 5. [32] An entropy on an HFE h is a real-valued function $E : H \to [0, 1]$, satisfying the following axiomatic requirements:

(E1) $E(h) = 0$, if and only if $h = \{0\}$ or $h = \{1\}$;
(E2) $E(h) = 1$, if and only if $h^i + h^{l-i+1} = 1$ for $i = 1, 2, \ldots, l$;
(E3) $E(h_1) \leq E(h_2)$, if $h_1^i \leq h_2^i$ for $h_2^i + h_2^{l-i+1} \leq 1$, or $h_1^i \geq h_2^i$ for $h_2^i + h_2^{l-i+1} \geq 1$, where h_1 and h_2 have the same length l obtained by repeating elements.
(E4) $E(h) = E(h^c)$.

Based on Definition 5, the following entropy measures are proposed:

$$\begin{aligned} E_1(h) =& \frac{1}{l(\sqrt{2}-1)}\sum_{i=1}^{l}\{\sin\frac{\pi(h^i + h^{l-i+1})}{4} \\ &+ \sin\frac{\pi(2 - h^i - h^{l-i+1})}{4} - 1\}, \end{aligned} \tag{14}$$

$$\begin{aligned} E_2(h) =& \frac{1}{l(\sqrt{2}-1)}\sum_{i=1}^{l}\{\cos\frac{\pi(h^i + h^{l-i+1})}{4} \\ &+ \cos\frac{\pi(2 - h^i - h^{l-i+1})}{4} - 1\}, \end{aligned} \tag{15}$$

$$\begin{aligned} E_3(h) =& -\frac{1}{l\ln 2}\sum_{i=1}^{l}\{\frac{h^i + h^{l-i+1}}{2}\ln\frac{h^i + h^{l-i+1}}{2} \\ &+ \frac{2 - h^i - h^{l-i+1}}{2} ln \frac{2 - h^i - h^{l-i+1})}{2}\} \end{aligned} \tag{16}$$

$$\begin{aligned} E_4(h) =& \frac{1}{l(2^{(1-s)t} - 1)}\sum_{i=1}^{l}\{((\frac{h^i + h^{l-i+1}}{2})^s \\ &+ (\frac{2 - h^i - h^{l-i+1}}{2})^s)^t - 1\}. \end{aligned} \tag{17}$$

Definition 5 generalizes the axiomatic definition of entropy for FSs [6] and aims to define the entropy measure of an HFE h based on the similarity degree of h and h^c. It is noted that the condition (E3) is too strong to discriminate the uncertain degrees of some HFEs, even though they are apparently different. While using (E3) of the Definition 3, the entropy values of more HFEs can be compared and this comparison is more coherent with people's intuition.

For example, suppose that the membership degree of one element to a set provided by one decision group is represented by the HFE $h_1 = \{0.1, 0.9\}$ and the membership degree provided by another decision group is represented by the HFE $h_2 = \{0.11\}$.

The HFE $h_1 = \{0.1, 0.9\}$ depicts the situation that some experts in the group provide the membership degree 0.1, and the other provide the membership degree 0.9. There are bigger disagreements among experts in this group. The information provided by the HFE $h_1 = \{0.1, 0.9\}$ involves hesitation and we are not sure whether the element belongs to the set or not. The HFE $h_2 = \{0.11\}$ implies that the element is belonging to the set with a degree 0.11. The information does not involve hesitation. Intuitively, the uncertain degree of h_1 is higher than h_2. But for the two HFEs $h_1 = \{0.1, 0.9\}$ and $h_2 = \{0.11, 0.11\}$, we have $0.11 > 0.1$ and $0.11 < 0.9$. So, from condition (E3) of the Definition 5, it is not possible to compare the entropies of the HFEs h_1 and h_2. However, taking into account (E3) of the proposed Definition 3, we have $E(h_1) > E(h_2)$ since $\theta(h_2) < \theta(h_1) \leq 0.5$ and $\eta(h_2) < \eta(h_1)$. This result accords with people's intuition.

On the other hand, using the condition (E3) of the Definition 5 to compare two HFEs, it is necessary to extend the shorter HFE to have the same length as the longer one. Any value in the shorter one can be added to extend it. The comparison results will be susceptible to the added elements. However, if (E3) of the proposed Definition 3 is used, it is not necessary to add any value to compare the HFEs.

For example, let $h_1 = \{0.2, 0.5\}$ and $h_2 = \{0.2, 0.3, 0.4\}$ be two HFEs. If h_1 is extended to $h_1 = \{0.2, 0.5, 0.5\}$, then $E(h_1) > E(h_2)$ using the condition (E3) of the Definition 5. But if it is extended to $h_1 = \{0.2, 0.2, 0.5\}$, then it is not possible to compare the two HFEs by this condition. Nevertheless, by using the condition (E3) of the Definition 5, since $\theta(h_2) < \theta(h_1) \leq 0.5$ and $\eta(h_2) < \eta(h_1)$, then $E(h_1) > E(h_2)$.

The entropy measures E_1, E_2, E_3 and E_4 defined by Equations 14–17 which consider the fuzziness of the HFEs are analyzed. In fact, for an HFE $h = \{h^1, h^2, \ldots, h^l\}$, the entropy measures change with the values of $\frac{h^i+h^{l-i+1}}{2}$. If $\frac{h^i+h^{l-i+1}}{2} < 0.5$, the bigger the value of $\frac{h^i+h^{l-i+1}}{2}$, then the bigger the value of $\theta(h)$ is, and furthermore the bigger the entropy is. If $\frac{h^i+h^{l-i+1}}{2} > 0.5$, the bigger the value of $\frac{h^i+h^{l-i+1}}{2}$, then the bigger the value of $\theta(h)$ is, and the smaller the entropy is. For any two HFEs h_1 and h_2 with the same length l, if $h_1^i + h_1^{l-i+1} = h_2^i + h_2^{l-i+1}$ for any $i = 1, 2, \ldots, l$, then $\theta(h_1) = \theta(h_2)$, thus $E(h_1) = E(h_2)$. In contrast to Equations 14–17, the entropy calculated by Equation 7 changes with the value of score function and the value of deviation function of an HFE. It considers not only the fuzziness of an HFE, but also its hesitation degree. Therefore, the entropy measure defined by Equation 7 has bigger ability to compare the uncertainties of HFEs.

On the other hand, Farhadinia [9] gave the following axiomatic definition of entropy measure based on the distance d proposed in [31] between two HFEs.

Definition 6. An entropy on an HFE h is a real-valued function $E: \ H \to [0, 1]$, satisfying the following axiomatic requirements:

(E1) $E(h) = 0$, if and only if $h = \{0\}$ or $h = \{1\}$;
(E2) $E(h) = 1$, if and only if $h = \{0.5\}$;
(E3) $E(h_1) \leq E(h_2)$, if $d(h, \{0.5\}) \geq d(h, \{0.5\})$
(E4) $E(h) = E(h^c)$.

Based on Definition 6, the following theorem is given in [9] to provide an approach to build a family of entropies for HFEs using a distance between HFEs.

Theorem 4. Let $Z : [0, 1] \to [0, 1]$ be a strictly monotone decreasing real function, and d be a distance between HFEs. Then $E_d(h) = \frac{2d(h,\{0.5\})-Z(1)}{Z(0)-Z(1)}$ is an entropy for the HFE h based on the corresponding distance d.

Xu and Xia [31] defined a variety of distance measures for HFSs that have been reviewed in Section 2. These distance measures can be used to calculate the distance between an HFE h and $\{0.5\}$. Therefore, in Theorem 4, if $Z(x) = 1 - x$ and d is defined by Equation 2 with $\lambda = 1$. Then

$$E_d(h) = 1-2d_{gh}(h, \{0.5\}) = 1-\frac{1}{l}\sum_{i=1}^{l} 2\left|h^i-0.5\right| \tag{18}$$

Note that Definition 6 is based on the distance d between the HFEs h and $\{0.5\}$, and it only considers the fuzziness of an HFE. Therefore, for any two HFEs that have the same distance to $\{0.5\}$, the entropy is equal by using the entropy formula E_d defined by Equation 18. For example, let $h_1 = \{0, 1\}$, $h_2 = \{0\}$ be two HFEs. The entropy obtained for them is $E_d(h_1) = E_d(h_2) = 0$. It is easy to see that the result is not consistent with our reasoning. In fact, $h_1 = \{0, 1\}$ represents that one expert is absolutely in favor and the other one is absolute opposition, and $h_2 = \{0\}$ represents that the two experts are absolute opposition. The uncertainty of information represented by $h_1 = \{0, 1\}$ should be the biggest. But using the Equation 18, we obtain $E_d(h_1) = E_d(h_2) = 0$, which is a shortcoming of the entropy measure defined in [9]. Moreover, from the above example, it can be seen that the entropy measure E_d defined in Theorem 4 does not meet the condition (E1) of the Definition 6.

4.2. Comparisons with examples

This subsection carries out a further comparison between the existing entropy formulas by means of an example.

Example 2. Let $h_1 = \{0.2, 0.4\}$, $h_2 = \{0.3, 0.5\}$, $h_3 = \{0.3, 0.4, 0.5\}$, $h_4 = \{0.4, 0.5\}$, $h_5 = \{0.3, 0.6\}$, $h_6 = \{0.3, 0.5, 0.6\}$, $h_7 = \{0.2, 0.5, 0.7\}$, $h_8 = \{0.4, 0.5, 0.6\}$, $h_9 = \{0, 1\}$, be nine HFEs.

The entropies of these HFEs are calculated by the entropy measures E_1, E_2, E_3, E_4, E_{g1}, E_{g2},E_{g3},E_{g4} and $E_d(\lambda = 1)$, respectively. The results are shown in Table 1.

From Table 1, we can see that entropy values calculated by $E_i(i = 1, 2, 3, 4)$ and $E_{gi}(i = 1, 2, 3, 4)$ get larger when the score function values $\theta(h)$ increase. But the entropies E_1, E_2, E_3 and E_4 cannot distinguish the uncertainty of HFEs that have the same score function values, since $E_i(h_6) = E_i(h_7)(i = 1, 2, 3, 4)$ and $E_i(h_8) = E_i(h_9)(i = 1, 2, 3, 4)$. The HFEs h_6 and h_7, h_8 and h_9 have the same score function values and different deviation values, respectively. The dispersion degree of elements in h_7 is greater than in h_6, and the dispersion degree of elements in h_9 is greater than in h_8. Obviously, h_7 is intuitively more uncertain than h_6, and h_9 is more uncertain than h_8. Using the proposed entropies $E_{gi}(i = 1, 2, 3, 4)$, it is obtained that $E_{gi}(h_6) < E_{gi}(h_7)(i = 1, 2, 3, 4)$, that it is consistent with human beings reasoning.

There is also some inconsistent results using E_d defined by Equation 18. The entropy $E_d(h_2) = E_d(h_3)$ and $E_d(h_4) > E_d(h_5)$, while $E_{gi}(h_3) < E_{gi}(h_2)$ and $E_{gi}(h_4) < E_{gi}(h_5)(i = 1, 2, 3, 4)$. Therefore, the results $E_d(h_2) = E_d(h_3)$ and $E_d(h_4) > E_d(h_5)$ are not consistent with our reasoning. In fact, h_2 and h_3, h_4 and h_5 have the same score function values respectively. The dispersion degree of elements in h_2 is greater than in h_3, and the dispersion degree of elements in h_5 is greater than in h_4. Obviously, h_2 is intuitively more uncertain than h_3 and h_5 is more uncertain than h_4, that it is the fact reflected by the entropies E_{gi} $(i = 1, 2, 3, 4)$.

From the above analysis, the entropy measures $E_{g_i}(i = 1, 2, 3, 4)$ are more effective to reflect hesitation and fuzziness of HFEs.

5. A method for multi-criteria decision-making based on entropy measures for HFEs

Entropy measures have been applied in many problems such as optimizing the distinguishability of input space partitioning and assessing the weights of experts or criteria in intuitionistic fuzzy decision-making [11, 25, 34]. In this section, a method is proposed to solve multi-criteria decision-making problems with unknown criterion weights [5]. The entropy measures for HFSs are used to determinate the criteria' weights and the idea of TOPSIS method is used to rank alternatives.

The multi-criteria decision-making problem which is considered in this paper can be represented as follows. There are m alternatives,

Table 1
The entropies of $h_i(i = 1, 2, \ldots, 9)$ by different entropy formulas

	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8	h_9
$\theta(h)$	0.3	0.4	0.4	0.45	0.45	0.47	0.47	0.5	0.5
$\eta(h)$	0.2	0.2	0.133	0.1	0.3	0.2	0.333	0.133	1
$E_1(h)$	0.833	0.958	0.958	0.989	0.989	0.993	0.993	1	1
$E_2(h)$	0.833	0.958	0.958	0.989	0.989	0.993	0.993	1	1
$E_3(h)$	0.881	0.971	0.971	0.993	0.993	0.995	0.995	1	1
$E_4(h)$	0.9165	0.9798	0.9798	0.995	0.995	0.997	0.997	1	1
$E_d(h)$	0.6	0.8	0.8	0.9	0.7	0.8	0.667	0.867	0
$E_{g1}(h)$	0.667	0.833	0.823	0.909	0.923	0.944	0.950	1	1
$E_{g2}(h)$	0.510	0.743	0.727	0.858	0.880	0.913	0.922	1	1
$E_{g3}(h)$	0.841	0.959	0.957	0.989	0.990	0.995	0.996	1	1
$E_{g4}(h)$	0.867	0.967	0.965	0.991	0.992	0.996	0.997	1	1

denoted by $X = \{x_1, x_2, \ldots, x_m\}$. Each alternative is assessed by means of n criteria, denoted by $C = \{C_1, C_2, \ldots, C_n\}$. Assume that the weights of the criteria $C_j (j = 1, 2, \ldots, n)$ are unknown. The characteristics of the alternative x_i in terms of the criterion C_j are represented by the following HFSs:

$$M_i = \{\langle C_j, h_{ij} \rangle | C_j \in C\}, \quad i = \{1, 2, \ldots, m\},$$

where h_{ij} is an HFE that indicates the degree in which the alternative M_i satisfies the criterion C_j. Espert's goal is to obtain a ranking of alternatives.

In practical multi-criteria decision-making problems, it is an important research topic to determine the weights of criteria. In the following, it is proposed a method to obtain the weights of criteria based on the proposed entropy measures according to experts' evaluation information, and a method to solve the above multi-criteria decision-making problem.

The multi-criteria decision-making approach contains three processes: (1) determinate the weights of the criteria; (2) derive the comprehensive evaluations of the alternatives; (3) rank the alternatives.

1. *Determination of the weights of the criteria*: the proposed entropy measures are used to compute the criteria weights. Let $C_j = \{\langle x_i, h_{ij} \rangle | x_i \in X\}, j = \{1, 2, \ldots, n\}$ be an HFS on the alternative set X, which includes the overall assessment values for all the alternatives $x_i (i = 1, 2, \ldots, m)$ under the criteria C_j. Therefore, the entropy $\overline{E}_j$ of C_j can be calculated by Equations 8–11, where $\overline{E}_j = \frac{1}{m} \sum_{i=1}^{m} E(h_{ij})$. $\overline{E}_j$ indicates the uncertainty degree of the assessment provided for the criterion C_j. During the practical decision-making process, we usually expect that the uncertainty degree of the assessment is as small as possible. Thus, if the entropy value $\overline{E}_j$ related to the criterion C_j is lower, we assign it a higher weight, and vice versa. Therefore, the criteria weights are defined as follows:

$$\omega_j = \frac{1 - \overline{E}_j}{n - \sum_{j=1}^{n} \overline{E}_j}, \quad j = \{1, 2, \ldots, n\} \quad (19)$$

2. *Deriving the comprehensive evaluations of the alternatives*: this process is divided into three steps.
   (a) Firstly it is necessary to find the positive-ideal solution and negative-ideal solution. Let J_1 and J_2 be the sets of benefit criteria and cost criteria in the criterion set C, respectively. Suppose that $H^+ = \{\langle C_j, h_j^+ \rangle | C_j \in C\}$ is the hesitant fuzzy positive-ideal solution, and $H^- = \{\langle C_j, h_j^- \rangle | C_j \in C\}$ is the hesitant fuzzy negative-ideal solution, where $h_j^+ = \{1\}$, $h_j^- = \{0\}$, $j \in J_1$ and $h_j^+ = \{0\}$, $h_j^- = \{1\}$, $j \in J_2$.
   (b) By using Equation 3, the distance between the alternative M_i and the positive-ideal solution or the negative-ideal solution can be computed:
   $$D^+(M_i) = d_{ghw}(M_i, H^+),$$
   $$D^-(M_i) = d_{ghw}(M_i, H^-),$$
   being $i = \{1, 2, \ldots m\}$.
   (c) The relative closeness degree $D(M_i)$ of the alternative M_i to the ideal solution is obtained as follows.
   $$D(M_i) = \frac{D^+(M_i)}{D^+(M_i) + D^-(M_i)}, \quad i = 1, 2, \ldots m.$$
3. *Ranking the alternatives*: the alternatives are ordered according to the relative closeness degrees.

The following steps show how to apply the multi-criteria decision making method.

Step 1. Calculate the criterion weights by Equation 19.

Step 2. Calculate the distances of each alternative to the positive-ideal solution $D^+(M_i)$, and the negative-ideal solution $D^-(M_i)$.

Step 3. Calculate the relative closeness degree $D(M_i)$ for each alternative.

Step 4. Rank the alternatives M_i according to the values of $D(M_i)(i = 1, 2, \ldots, m)$ in ascending order, and the smaller the value of $D(M_i)$, the better the alternative M_i .

6. Illustrative example

A case study concerning the health-care waste management is employed to illustrate the efficiency

Table 2
Criteria to evaluate a telecommunications service

Criterion	Description of criterion
Price C_1	How the company is satisfied with the price, which will be paid for the telecommunications service
Quality C_2	What level the telecommunications service can reach
Service C_3	The maintenance and repair
Safeguard C_4	The reliability of information protection

Table 3
Assessments over the alternatives and criteria

E_{g1}	C_1	C_2	C_3	C_4	E_{g2}	C_1	C_2	C_3	C_4
M_1	0.9091	0.7273	0.7273	0.5455	M_1	0.8578	0.5873	0.5873	0.3572
M_2	0.6471	0.5455	0.5455	0.9091	M_2	0.4814	0.3572	0.3572	0.8578
M_3	0.8235	0.9091	0.8235	0.7273	M_3	0.7273	0.8578	0.7273	0.5873
M_4	0.3636	0.7273	0.9143	0.6471	M_4	0.19	0.5873	0.8659	0.4814
M_5	0.6471	0.9143	0.4706	0.8235	M_5	0.4814	0.8659	0.2862	0.7273
E_{g3}	C_1	C_2	C_3	C_4	E_{g4}	C_1	C_2	C_3	C_4
M_1	0.9888	0.9009	0.9009	0.7337	M_1	0.9909	0.9182	0.9182	0.7727
M_2	0.8315	0.7337	0.7337	0.9888	M_2	0.8588	0.7727	0.7727	0.9909
M_3	0.9568	0.9888	0.9568	0.9009	M_3	0.9647	0.9909	0.9647	0.9182
M_4	0.5036	0.9009	0.9894	0.8315	M_4	0.5545	0.9182	0.9914	0.8588
M_5	0.8315	0.9894	0.6363	0.9568	M_5	0.8588	0.9914	0.6824	0.9647

Table 4
Entropies and weights of the criteria

	$\overline{E}_1$	$\overline{E}_2$	$\overline{E}_3$	$\overline{E}_4$		w_1	w_2	w_3	w_4
E_{g1}	0.6781	0.7647	0.6962	0.7305	E_{g1}	0.2848	0.2082	0.2687	0.2384
E_{g2}	0.5476	0.6511	0.5648	0.6022	E_{g2}	0.2768	0.2135	0.2663	0.2434
E_{g3}	0.8224	0.9028	0.8434	0.8824	E_{g3}	0.3234	0.1771	0.2852	0.2143
E_{g4}	0.8456	0.9183	0.8659	0.9011	E_{g4}	0.3292	0.1742	0.2858	0.2108

Table 5
Distances and relative closeness degrees for each alternative

E_{g1}	$D^-(M_i)$	$D^+(M_i)$	$D(M_i)$	E_{g2}	$D^-(M_i)$	$D^+(M_i)$	$D(M_i)$
M_1	0.4737	0.5773	0.5493	M_1	0.4736	0.578	0.5497
M_2	0.678	0.3623	0.3482	M_2	0.6771	0.3636	0.3494
M_3	0.6064	0.4058	0.4009	M_3	0.6064	0.4058	0.4009
M_4	0.613	0.486	0.4422	M_4	0.6101	0.4882	0.4445
M_5	0.6842	0.3529	0.3403	M_5	0.6828	0.3545	0.3417
E_{g3}	$D^-(M_i)$	$D^+(M_i)$	$D(M_i)$	E_{g4}	$D^-(M_i)$	$D^+(M_i)$	$D(M_i)$
M_1	0.4727	0.5751	0.5488	M_1	0.4731	0.5743	0.5483
M_2	0.6825	0.3559	0.3428	M_2	0.6831	0.355	0.342
M_3	0.6068	0.4056	0.4006	M_3	0.6068	0.4057	0.4007
M_4	0.6259	0.4759	0.4319	M_4	0.6281	0.4741	0.4301
M_5	0.6924	0.344	0.3319	M_5	0.6932	0.3431	0.3311

of the multi-criteria decision making method proposed in the previous section.

Nowadays, the competition among telecommunications services is increasing and it is much more difficult for SMEs (Small and Medium-sized Enterprises) to choose a suitable telecommunications service to improve their business operations, since ample resources can be a big obstacle. Suppose that a SME has to select the best telecommunications service provider to improve its benefits. There are five possible alternatives: provider 1 (M_1), provider 2 (M_2), provider 3 (M_3), provider 4 (M_4) and provider 5 (M_5). Based on the society research, four major criteria are considered to evaluate these five telecommunications service providers. These criteria are: The satisfaction of price (C_1), Quality (C_2), Service (C_3), and Safeguard (C_4). A detailed description of such criteria is given in Table 2.

Let us suppose a decision organization with five experts authorized to assess the satisfactory degree of an alternative with respect to a criterion, which is represented by an HFE. The evaluations of the five possible alternatives $M_i (i = 1, 2, \ldots, 5)$ under the above four criteria can be represented by the following HFSs:

$M_1 = \{\langle C_1, \{0.5, 0.6\}\rangle, \langle C_2, \{0.6, 0.7\}\rangle, \langle C_3, \{0.3, 0.4\}\rangle, \langle C_4, \{0.2, 0.3\}\rangle\}$,
$M_2 = \{\langle C_1, \{0.6, 0.7, 0.8\}\rangle, \langle C_2, \{0.7, 0.8\}\rangle, \langle C_3, \{0.7, 0.8\}\rangle, \langle C_4, \{0.4, 0.5\}\rangle\}$,
$M_3 = \{\langle C_1, \{0.5, 0.6, 0.7\}\rangle, \langle C_2, \{0.5, 0.6\}\rangle, \langle C_3, \{0.5, 0.6, 0.7\}\rangle, \langle C_4, \{0.6, 0.7\}\rangle\}$,
$M_4 = \{\langle C_1, \{0.8, 0.9\}\rangle, \langle C_2, \{0.6, 0.7\}\rangle, \langle C_3, \{0.3, 0.4, 0.5, 0.6\}\rangle, \langle C_4, \{0.2, 0.3, 0.4\}\rangle\}$,
$M_5 = \{\langle C_1, \{0.6, 0.7, 0.8\}\rangle, \langle C_2, \{0.4, 0.5, 0.6, 0.7\}\rangle, \langle C_3, \{0.7, 0.8, 0.9\}\rangle, \langle C_4, \{0.5, 0.6, 0.7\}\rangle\}$.

The multi-criteria decision making approach proposed in Section 5 will be used to get the most desirable alternative(s).

Step 1. Determine the criterion weights.
With the entropy measures E_{g1}, E_{g2}, E_{g3} and E_{g4} defined by Equations 8–11, are calculated the entropies of HFEs, which construct the entropy matrices shown in Table 3, respectively.
According to each entropy matrix calculated by E_{gi} and Equation 19, the entropies $\overline{E}_i$ and the weights w_i of criteria $C_i (i = 1, 2, 3, 4)$ are calculated and shown in Table 4.

Step 2. Calculate the distance $D^+(M_i)$ and $D^-(M_i)$ between each alternative and the positive-ideal and negative-ideal solution, respectively. Afterwards, the relative closeness degrees $D(M_i)$, for each alternative is obtained (see Table 5).

Step 3. Taking into account the relative closeness degree obtained for each alternative $D(M_i)(i = 1, 2, \ldots, m)$, the ranking of alternatives obtained by using different entropy measures $E_{gi}(i = 1, 2, 3, 4)$ is as follows:

$$M_1 \succ M_4 \succ M_3 \succ M_2 \succ M_5.$$

In the above calculation process, we adopt the entropy formulas E_{g1}, E_{g2}, E_{g3} and E_{g4} to calculate the criteria weights and the relative closeness degrees of alternatives, respectively. From Table 4, it is easy to see that criteria weights obtained by each entropy measure are different, but the ranking of these weights are the same, that is, $w_1 > w_3 > w_4 > w_2$. Under different entropy measures, the relative closeness degrees of alternatives are different, but the gaps between these values are very small, therefore the ranking of alternatives is the same.

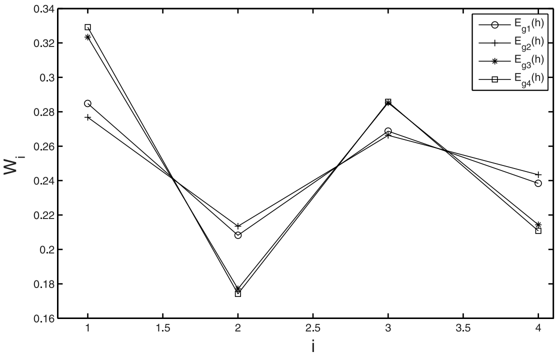

Fig. 2. Comparisons of weights under E_{gi}.

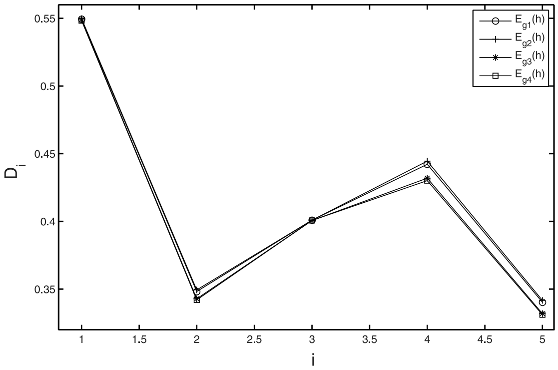

Fig. 3. Comparisons of relative closeness degrees under E_{gi}.

The comparisons of weights and relative closeness degrees under the four entropy measures are shown in Figs. 2 and 3.

7. Conclusions

An HFS, whose membership is represented by a set of possible values, is more suitable to represent the uncertain information when experts hesitate among several values. In this paper, the entropy measures (being important topics of information measures) of HFEs and HFSs have been studied. An axiomatic definition and a concrete entropy formula for HFEs have been proposed. Then a family of entropy measures based on the entropy measure aforementioned has been presented. These entropy measures have been compared with the existing ones. The comparison results reflect that the proposed axiomatic definitions and entropy measures can depict both fuzziness and hesitation of HFEs and HFSs, and they are able to compare HFEs when the existing ones cannot. Finally, the proposed entropy measures have been used to determinate the criteria weights and a multi-criteria decision making approach has been developed to deal with hesitant fuzzy information. In the future, we will further investigate the entropy theory on the hesitant fuzzy linguistic term sets and its application in decision making.

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